LECTURES ON CORPORATE FINANCE

Second Edition

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Second Edition

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To Betty, Majanka and Frederik

To Ingeborg Helene and Arne Tobias
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Introductory Remarks

Scope

This course of lectures introduces students to elementary concepts of corporate finance using a more systematic approach than is generally found in other textbooks. Axioms are first highlighted and the implications of these important concepts studied afterwards. These implications are used to answer questions about corporate finance, including issues related to derivatives pricing, state-price probabilities, dynamic hedging, dividends, capital structure decisions, and risk and incentive management.

The main thing to note about this book compared to more standard texts in corporate finance is the level of abstraction. We are arguing in an abstract manner to make the unifying themes, represented by the axioms, clear. But this does not mean that we are using advanced mathematics. While we are not afraid of using mathematical expressions where it will simplify understanding, the emphasis is on basic algebra. No advanced calculus or stochastic processes is used. We provide an extensive set of examples in this book. Most of them are relatively simplistic, they are used to clarify a single point.

We are on purpose not trying to be encyclopedic in our coverage of finance. This book is mainly on principles, little about the nitty-gritty of institutions, in which many finance text abounds.

In order to understand the concepts in this book it is essential to work with numerical problems. End of chapter problems are provided for most chapters. Each problem has an indicated level of difficulty, ranking from 1 (simple) to 10 (very hard). If there is a lot of work involved with a problem, this will also push it up the scale.
Acknowledgment

Numerous students have gone through the first edition of the book, and we owe much to them for pointing out typos and suggesting improvements in the exposition. We would also like to thank Steinar Ekern for his constructive criticism of our handling of capital budgeting under CAPM, and Chester Spatt for suggesting a simple example that illustrates the power of the usage of state price probabilities.

A Roadmap (Where are we going?)

Introductory chapters, setting the stage. We present some of the "axioms" that we rely on in the later analysis. Like all axioms, they sound reasonable.

- "Axioms" of Modern Corporate Finance.
- On Value Additivity.
  The finance equivalent of Lavoisier's law. An implication: The value of a firm.
- On The Efficient Markets Hypothesis.
  Loosely stated, it claims that securities prices should not be "too" predictable, because otherwise there is money on the table.

Basics chapters, covered in all finance books.

- Present Value.
  Prices of future cash flows are expressed in terms of interest rates. To value a stream of cash flows, you "discount" using these interest rates. The terminology is strange, but the principles are no different from what grocery stores use.
- Competing Valuation Methods.
  People have been proposing lots of alternative valuation methods, some of which are merely restatements of standard present value analysis, others deliver investment decisions that range from sometimes the same (as present value) to always incorrect, at least if you buy the axioms of modern corporate finance.
- Valuation Under Uncertainty: The CAPM.
  One popular way to distinguish between classes of risky future cash flows is to compute "betas" (covariances with the market portfolio). Cashflows get
discounted on the basis of a class-specific, risk-adjusted discount rate. The idea comes from a simple, intuitively appealing, equilibrium asset pricing model, the Capital Asset Pricing Model (CAPM). Even far more elaborate (and realistic) models basically deliver the same insights as the CAPM, namely that only covariance with "aggregate risk" is priced. To put it differently: one is not rewarded for all the uncertainty that an investment project or security carries, only the "systematic" uncertainty.

Multiperiod pricing and derivatives.

- Valuing dated, risky cash flows based on an enumeration of states.

The CAPM or APT does not work well to value such cash flows, because they are not symmetric in good and bad states. So, we need a novel approach.

The valuation approach for this part is incredibly simple. We divide the world into possible future "states." We then price cash flows in each state separately, using "state-price probabilities," before we add everything together, using the axiom of value additivity.

- Basics of derivatives.

We will start with some basic facts about the quintessential derivative products, call and put options.

- Valuing derivatives

Many cash flows derive from the future value of some underlying asset, and, hence are really derivative products. We will look how the state price approach generates sensible prices for call options and equity, including a real-life example.

- Where To Get State-Price Probabilities?

It is often obvious what the states are. If not, the most straightforward procedure obtains states from the average and the volatility of the payoff on the underlying asset. Since any derivative (including the underlying asset itself) should have the same implicit state-price probabilities, we can back out the state-price probabilities from their prices.

- The Dynamic Hedge Argument.

Why are these state-price probabilities the same for all derivatives written on the same underlying asset? As we shall see, it is because there would otherwise be free lunches... We do not really believe such a thing exists, do we? A variation of the arguments leads us to discover the way that the
pricing formula was historically developed by Cox, Ross, and Rubinstein (1979) who called it the binomial option pricing model.

- More on the binomial option pricing model.
  The power of the binomial approach becomes clear when we increase the number of periods.

- An Application: Pricing Corporate Bonds.
  The best way to understand the derivatives valuation approach is to look at applications. We already looked at call options, equity and warrants. We will get more sophisticated, and consider "fixed income" securities that may not seem to be derivatives. We will see how the resulting valuation gives the shareholders the incentive to "screw" the bondholders by changing the risk of the company.

Corporate Finance

- Are Capital Structure Decisions Relevant?
  Until now, we have been pricing corporate liabilities (equity, warrants, debt) as derivatives written on the assets of the firm. Their value derives from the value of the firm. That is fine, but we have been taking the latter to be exogenous. Is the value of the firm (as defined in Chapter 2) really fixed? Can we not change it by changing the financing mix (ratio of debt to equity)? F. Modigliani and M. Miller convincingly argue: no.

- Maybe They Are?
  Third-party creditors on the corporate scene, like lawyers and tax collectors, will overturn the celebrated "Modigliani-Miller" irrelevance result.

- Valuation Of Projects Financed Partly With Debt.
  We will contrast the derivatives valuation approach with others that have been suggested to tackle the problem.

- And What About Dividends?
  Should a company pay dividends? Sounds awful, because the only to benefit is the tax collector! Why, then, do so many firms pay dividends?

Risk And Incentive Management

- Since derivatives valuation is based on hedging, it suggests plenty of ways to manage risk. That is nice for bondholders and shareholders alike, but also for employees! In particular, it suggests a way for Sun Microsystems
to keep their engineers in the face of potential redundancy when Microsoft manages to kill Java...

Insights.

• We summarize the insights of the book.

Longer Examples.

• To make sure we understand what we have been doing, here are some specific examples.
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Part I

Introduction to Finance
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Chapter 1

Finance

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1.1 What is Finance?

Finance is study of the valuation and management of risk.

There are two components to risk.

1. The time of its revelation.

2. The nature of its randomness.

We start with the "timing" problem by considering valuation and management of risk free cash flows at different points in time.

The second problem is more tricky. How can one distinguish between different classes, or categories, of risky cash flows? We all know how to distinguish between apples and pears, but what is the equivalent for risky cashflows?

1.2 Corporate Finance

This book is about financial issues connected with corporations. Typical definitions of corporations deal with the legalities of defining the ownership and control. But we will largely abstract from these. From a finance point of view, the corporation is a bundle of risky cash flows. To value a corporation, we disentangle
the components of the risky cash flows (including the date of revelation), value these separately, and, finally, apply value additivity to sum all the components.

For our purposes, managing a company is largely playing around with the bundling of the risky components of the corporation. For example, reshuffling of cash flows in time to postpone tax payments, or understanding how bundling creates incentives for participants, such as management, bondholders and equity owners.

1.3 Financial Markets

For our purposes, financial markets can be thought of as supermarkets for risky cash flows. Unlike regular supermarkets where you shop among products on the shelves, financial supermarkets work as organized exchanges, where financial securities are bought and sold in continuous auctions. Financial securities are best though of as packages of cash flows, and they come in all sorts. Let us look at some examples.

Most governments need to finance their deficits. To do so they issue government debt. The debt can be short term, in which case we call it a Treasury Bill, which will promise a given cashflow sometime within the next year. A treasury bill is the typical example of a risk free security, there is no uncertainty about the future payments. If government debt is long term, over a year, it is typically issued as government bonds, with annual payments of coupons and a repayment of the face value at the bond maturity. Since governments can always print money, there is no uncertainty about whether you get your money back when you hold a long term government bond. But government bonds are still not as risk free as treasury bills. The reason is that there is always uncertainty about the future worth of a dollar (or pound, or mark) due to inflation.

A more risky security is corporate equity, or stocks. A corporate equity gives the owner the right to a dividend from the corporation. The dividend is a function of the profitability of the corporation. Since this profitability is quite variable for most companies, the cashflows from a stock will be risky. Historically, though, at least in the U.S, the return from holding equity has been on average much higher than the return from holding government debt.

Some of the largest financial markets are markets in derivatives, securities whose payoff depend on the price of some other security, or even on the prices of real (as opposed to financial) goods. Futures markets are markets where one can fix a price today for a future delivery of some good. Options markets are markets where one can fix a price today for a future contingent delivery of some good.
We usually make a distinction between the primary and the secondary market. The primary market is at the issue of a security. Treasury securities are often issued to the general public by an auction where anybody can send in bids. This is then the primary market for treasury securities. When a corporation issues equities for the first time, the Initial Public Offering, this is the primary market for equities.

When securities are traded after they have been issued (in the primary market), they are said to be traded in the secondary market. In terms of volume and value, the secondary market dwarfs the primary market. Many do not understand why secondary markets are important, because they only seem to be “zero sum games.” (If somebody makes a gain buying a stock the seller must be a loser.) These people miss some important services that financial markets provide:

- Hedging (risk insurance).
- Intertemporal matching of liquidity needs.
- Price signals to primary market.

Relative to the amounts being bought and sold in financial markets, the costs of transacting are small. (Hey, all you are doing is shuffling paper around.) The costs are not zero (How do you think stockbrokers survive?), but they are closer to the economist’s definition of a perfect market than most other markets. It is this that makes it justifiable for us to make an assumption of perfect capital markets, markets where it is costless to transact. Their existence would in practice lead to extreme investment strategies, like: “Never realize any capital gains until you die, realize losses as soon as they occur, shield dividends from taxation by borrowing money.” When we use them in our models, they are best thought of as reasonable approximations to the “true” market.
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Chapter 2

Axioms of Modern Corporate Finance

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In this chapter we list the axioms we use in the remainder of the book.

2.1 The Axioms

2.1.1 Financial Markets are Competitive

Most financial markets are examples of the typical competitive market of economists, that is, participants in financial markets (individuals, corporations) must take prices as given.

Example

The current price of a contract for one dollar to be delivered in 12 months is 0.90. Even if a corporation sells one million units of this contract, it does not affect its price, the company will get

\[ 0.90 \cdot 1,000,000 = 900,000. \]

No more, no less.
2.1.2 Value Additivity

The price of a basket of financial contracts is equal to the sum of the prices of each individual contract times the quantities.

Example

An 8% Treasury bond with one year till maturity and $1,000 face value is really a basket of units of two financial contracts. One to deliver one dollar in 6 months (when half of the coupon is paid), and another to deliver one dollar in 12 months. The basket contains 40 units of the first contract, and 1,040 units of the second contract. If \( P_1 = 0.95 \) is the price of the first contract, and \( P_2 = 0.90 \) is the price of the second contract, then the value of the bond is:

\[
P_{140} + P_21,040 = 0.95 \cdot 40 + 0.90 \cdot 1,040 = 974,
\]

that is, 97.4% of the face value of the bond.

Do you know any supermarket that violates the rule of value additivity when ringing up a customer? Financial (super)markets don’t either!

2.1.3 No Free Lunches

It should be impossible to sell for a positive price a portfolio which has zero payoff for sure at all future times. This is also called the “no arbitrage” assumption.

The no arbitrage assumption is intimately connected to the value additivity assumption, since a violation of value additivity will also be a violation of the no free lunch assumption. They are still different principles. One says how to compute portfolio value; the other prescribes how to price a particular portfolio.

2.1.4 The Efficient Markets Hypothesis

Prices in financial markets will at all times reflect unbiased beliefs about the future.

An efficient market is really never surprised. Bad news happens with a certain frequency, and the market knows this. The market just does not know how the occurrence of bad news is distributed over time. In other words: markets know the frequency with which uncertain events occur, such as default, dividend increases, democratic party presidents and magnitude 7+ earthquakes in Southern California.

Of course, the precision of the market’s beliefs depends on the available information, but it is never systematically biased. Any systematic biases will be used to make profits above those justified by the risk of a strategy.
Chapter 3

On Value Additivity

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3.1 Value Additivity

There is a principle that no grocery store will ever want to violate:

The price of a basket of goods (apples, pears,...) equals the sum of the prices of the individual goods.

In finance we merely replace the goods with financial assets, such as

- cash flows to be delivered tomorrow, in one week, one month, ...
- portfolios of equities with random values one year from now.
- options.
- ...

3.2 Application to Corporate Finance: The Value of the Firm

A corporation holds a certain number of assets. Applying value additivity, the value of those assets is simply the sum of the values of the components. That is
On Value Additivity

called the value of the firm. The firm is held by a number of different creditors, such as equityholders, bondholders, banks and so forth. The value of the creditor's holdings, called liabilities, must by value additivity and no free lunches add up to the value of the firm.

We define the value of a firm as the price for which one could sell the stream of cash flows that the assets of the firm generates for the traditional creditors. Traditional creditors are: bondholders, warrantholders, shareholders, banks.

By the value additivity axiom and no free lunches,

The value of the firm is always equal to the sum of the values of the company's liabilities to the traditional creditors.

For example, if the company's liabilities consist of equity with value $E$ and debt with value $B$, and the firm value equals $V$, then:

$$V = E + B.$$ 

The summation on the RHS is justified by value additivity, the equality by "no free lunches."

But some care is advised here. The value of the firm is not necessarily equal to the price of the assets of the company. This because some of the cash flows go to third parties, in the form of taxes, lawyers and accountants fees and also because it may be advantageous to keep the company alive rather than selling it. The "live firm" may be more valuable e.g. because of its possibility for reducing future taxes. "Selling the stream of cash flows of the firm" does thus not necessarily mean "liquidation." Another thing to note is that the value of the firm almost never equals the book value of the firm's assets.

---

Example

The assets of a firm generate a perpetual, riskfree, after-tax cash flow of $10 per year. The riskfree rate is 5% p.a. The value of the firm equals

$$\sum_{t=1}^{\infty} \frac{10}{(1 + 0.05)^t} = \frac{10}{0.05} = \$200.$$ 

---

3.3 A Couple of Brain Teasers

Let us list a couple of issues that may look like violations of the axioms of value additivity and no free lunches on first glance. The reader should return to these after having gone through the book and see if she can understand why these may not be violations after all.
Closed-end mutual funds are regular companies whose sole purpose it is to invest in other shares. Usually, the shares (of equity) in closed-end mutual funds trade at a discount to the net asset value (the value of the shares the fund holds, less any liabilities to bondholders and/or banks). The discount can be as high as 25%. Is this a violation of value additivity and no free lunch?

Options give the right to purchase (call) or sell (put) an underlying asset at a pre-determined price (the strike price) during a particular period. Merton proved in 1973 that the value of an option on a portfolio of assets (think of it as equity in a firm with multiple subsidiaries) can never exceed, and will usually be lower than, the sum of the values of options on the component assets of the portfolio (think of it as the combined equity in each of the subsidiaries of the firm). How can it ever be lower? That seems to violate value additivity and no free lunch, but it does not really. Merton’s result has been used to explain why the stock price of companies that disintegrate (or spin off subsidiaries) increases.

Warren Buffett claims to have made a living out of violations of value additivity and no free lunch.
Problems

3.1 *Ketchup* [2]
As an empirical investigation, check your local supermarket. Does 2 ketchup bottles of 0.5 litres cost the same as one ketchup bottle of 1 liter? What does this tell you about value additivity in financial markets?

3.2 *Milk* [2]
Why is skimmed milk always cheaper than regular milk even if it is healthier?
Chapter 4

On the Efficient Markets Hypothesis

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4.1 The Idea

A systematic study of the behavior of securities prices started only in the late 50s, although Bachelier made a substantial contribution in 1900, developing, among other things, what we now know as the Wiener process as a description of stock prices. The basic intuition of efficient markets hypothesis is that to a large degree,
one cannot predict security prices. In fact, if there is "too much" predictability, speculators move in and profit. Their actions will make the market more "efficient," that is, less predictable. But what is "too much" predictability?

4.2 Traditional Formulation of the Efficient Markets Hypothesis

In the late sixties a number of theorists of finance, including Eugene Fama, tried to formalize the basic intuition about "little" predictability into The Efficient Markets Hypothesis: (Fama, 1970)

"Information is always correctly reflected in securities prices"

The problem with this statement is that it is ambiguous. What "information" are we talking about? What does "correctly reflected" mean?

4.3 What Information?

The way we specify information makes clear we are really thinking about how we can empirically test the efficient markets hypothesis. To make concrete the notion of information, we usually use three different "information sets," illustrated in figure 4.1.

---

**Figure 4.1 Information sets: Possible Definitions of the Market Information**

- All (private and public) Information
- Public Information
- Past Prices
4.4 What Does “Correctly Reflected” Mean?

We start with an information set we actually have some hope of gathering, the set of all past prices of financial assets. If knowing only this much information we are able to predict future prices, this is said to be a violation of weak form efficiency. This is also how one can test this form of market efficiency.

Example
The presence of autocorrelation (negative price movements are more likely to be followed by positive than negative price movements) in stock market returns will violate (some formulations of) weak form efficiency.

The second information set we consider is the set of all publicly available information. Note that past prices is part of this, so past prices is a subset of all publicly available information. If financial prices reflect all publicly available information, prices are said to satisfy semi–strong form efficiency. Here it is easy to see how one would go about testing this particular version of the efficient markets hypothesis: Do market prices react correctly to new information when it becomes public?

Example
When a corporation issues its quarterly earnings announcement and the earnings are twice what the market expected, the corporation’s stock price should rise.

The final information set is the set of all information. If markets were perfectly informed, and prices reflected all this information, prices would be strong form efficient. Under strong form efficiency, prices also reflect private (“insider”) information. The reasoning is that if an insider attempts to trade on her information, prices will move against her, and, because of this, reflect her information. Trading will reveal even inside information.

There is lots of evidence in favor of weak form and semi-strong form efficiency. There is solid evidence against strong form efficiency, markets can not perfectly read the information in trades.

4.4 What Does “Correctly Reflected” Mean?

It means that markets use the information available to them in the best possible way to generate its assessment of the current value of a financial security. Here is how this is traditionally interpreted.

4.4.1 Rational Learning

If learning is rational, it must satisfy the rules of conditional probability.

\[
p_{i,t} = E[p_{i,t+1} | \text{Information}_t]
\] (4.1)
Here $p_{i,t}$ is the price of security $i$ at time $t$. For simplicity, this security does not pay dividends. Information$_t$ is the information available at time $t$, and is used in generating the conditional expectation above.

### 4.4.2 Unbiased Beliefs

Also, there can be no systematic bias to the way expectations are formed, they must be based on the actual frequencies with which uncertain events happen.

$$p_{i,t} = E^{\text{TrueFreq}}[p_{i,t+1} | \text{Information}_t].$$  \hspace{1cm} (4.2)

where $E^{\text{TrueFreq}[\cdot]}$ is the expectation based on the actual frequencies. Notice how finance really takes an anthropomorphic view of the market.

### 4.4.3 Adding Compensation for Waiting and Risk

But there must be a return from holding a financial asset, otherwise why should you? The previous formal statements of the EMH are too simple. They fail to take into account that for anybody to be willing to hold a financial security, they must be offered an inducement to holding the security until the payoff is realized. Why hold on to a security with zero expected return if you earn 5% in the bank? We therefore refine the basic statement above. If a security was risk free, we could discount its next period price at the risk free interest rate $r_{f,t}$ to arrive at todays price.

$$p_{i,t} = \frac{1}{1 + r_{f,t}} E^{\text{TrueFreq}}[p_{i,t+1} | \text{Information}_t].$$

But most securities are not risk free, and we need to offer something more to compensate for the risk involved in holding it. We term this a risk premium, and use $\rho_{i,t}$ for the risk premium for security $i$ in period $t$.

$$p_{i,t} = \frac{1}{1 + r_{f,t} + \rho_{i,t}} E^{\text{TrueFreq}}[p_{i,t+1} | \text{Information}_t].$$

If we let $r_{i,t+1}$ denote the return on asset $i$ during the period from $t$ to $t + 1$, calculated as

$$r_{i,t+1} = \frac{p_{i,t+1} - p_{i,t}}{p_{i,t}},$$

we can formulate the main implication of the Efficient Markets Hypothesis as follows:

$$E^{\text{TrueFreq}}[r_{i,t+1} - r_{f,t} | \text{Information}_t] = \rho_{i,t}. \hspace{1cm} (4.3)$$

or in words:
Average Returns Only Reflect Compensation for Risk.

Note the implications of this. Price changes cannot be predicted beyond the compensation $p_{1,t}$ for risk. But then the EMH does allow prices to be "somewhat" predictable, and this predictability is time varying if the risk premium changes over time.

Also note that news does not affect average (expected) returns. This may seem paradoxical, since news does affect prices. But a bit of reflection on timing should put this right. The statement is about expected returns not changing. This expectation is formed just after the news have hit the market, at which time prices adjusted to the news. Bad news will push prices down, making the historical (ex post) returns go down, but the expectations based on the current price are not affected. News won't affect historical average returns if it is repetitive. In probability theory, this is referred to as stationarity. It means that good and bad news occur with a certain frequency. The market is supposed to know this frequency (see (4.2)), so the good and bad news washes out in the long run. In empirical studies, stationarity is invariably assumed. Notice that an efficient market is therefore never surprised about the nature of news (good and bad events happen with a certain frequency, and the market knows this). It is at most surprised about the timing: It would rather have bad news occur when everybody is rich.

4.5 Empirical Evidence

There is a tremendous amount of empirical work in finance that relates to the Efficient Markets Hypothesis, and it would take way too long to go into here. Surveys can be found in Fama (1991) and Leroy (1989).

4.6 Some Implications of the Efficient Markets Hypothesis

4.6.1 One Cannot Time the Market

At times, it may appear cheaper to wait to issue, say, corporate bonds, at a future date when coupons are expected to be lower. You could try to get short-term bridge financing. But that will carry a high opportunity cost, reflected in the high short-term interest rate.

Of course, the market may not know that coupon rates will decrease soon. But that very possibility contradicts market efficiency.
4.6.2 Average Returns in Excess of the Risk Free Rate are Solely Determined by Risk

This statement may seem rather paradoxical, because market efficiency also implies that securities prices must react to news about future cash flows. If such news has continuously been positive lately, would you not expect the average realized return to be exceptionally high as well?

4.6.3 Expected Returns Can Vary Over Time

When average excess returns vary over time in predictable ways, it is because risk varies in predictable ways.

References

4.6 Some Implications of the Efficient Markets Hypothesis

Problems

4.1 Interest Rates [2]
Consider the following statement.

*Long term interest rates are at record highs. Most companies therefore find it cheaper to finance with common stock or relatively inexpensive short-term bank loans.*

What does the Efficient Market Hypothesis have to say about the correctness of this?

4.2 Semistrong [3]
Can you expect to earn excess returns if you make trades based on your broker's information about record earnings for a stock, rumors about a merger of a firm, or yesterday's announcement of a successful test of a new product, if the market is semi-strong form efficient?

4.3 UPS [3]
On 1/10/85, the following announcement was made: "Early today the Justice Department reached a decision in the UPC case. UPC has been found guilty of discriminatory practices in hiring. For the next five years, UPC must pay $2 million each year to a fund representing victims of UPC policies." Should investors not buy UPC stock after the announcement because the litigation will cause an abnormally low rate of return over the next five years?

4.4 Management [3]
Your broker claims that well-managed firms are not necessarily more profitable investment opportunities than firms with an average management. She cites an empirical study where 17 well-managed firms and a control group of 17 average firms were followed for 8 years after the former were reported in the press to be "excelling" as far as management is concerned. Is this evidence that the stock market does not recognize good management?

4.5 TTC [3]
TTC has released this quarter's earning report. It states that it changed how it accounts for inventory. The change does not change taxes, but the resulting earnings are 20% higher than what it would have been under the old accounting system. There is no other surprises in the earnings report.

1. Would the stock price now jump on the release of this earnings report?
4.6 Investing? [3]
Does the following statement make sense in view of the Efficient Markets Hypothesis (EMH)?

The Japanese economy has deep structural problems, which the Japanese seem reluctant to overcome. We do not see any major change in this situation over the next two to three years. Hence, we advise against investing in the Tokyo stock market, because we expect returns to be below average for the next two to three years.
Part II

Basic Finance
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Chapter 5

Present Value

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5.1 Definition of Present Value

A basic unit of account in finance is a set of future cash flows. There are two important properties of these cash flows. One, the amounts. Two, the dates at which the amounts are paid.

For example, a US Government Treasury bill is a promise to pay USD 1000 at a certain future date. This is the typical example of a risk free security, one with no uncertainty as to both the amount of and timing of cash flow. As another example, consider an oil company about to start drilling for oil in a new area. The oil company is facing uncertainty at several levels. There is uncertainty about
whether the oil company actually locates oil. Even if they locate oil, there is a lot of uncertainty to what price they can sell the oil. In this case it is very hard to find the future cash flows, about the best one can do is to estimate the expected future cash flows. Alternatively one can consider contingent future cash flows.

We will for the rest of the chapter concentrate on the valuation of a sequence of certain future cash flows. The valuation of future risky cash flows is the topic of later chapters. We use the symbol $X_t$ for the amount $X$ to be paid at a future date $t$, and we want to value a set of future cash flows:

Given the set of dated cash flows, we define its Present Value (PV) as its value today. More precisely, it is the cost to obtain the same stream of cash flows in the market.

Evaluating the PV is simplified by using the axiom of value additivity, since we can then split the problem into summing the values of the individual dated cash flows. The problem is then reduced to finding the value today of a cash flow $X_t$ at some future date $t$. To do this we use the set of prices $P_t$ today of receiving one dollar at time $t$ in the future.

The PV of the entire stream is then:

$$PV = \sum_{t=1}^{T} P_t X_t.$$
Value additivity buys you other nice things. For example, we can split the evaluation of the present value into several steps. If we let $PV_0$ be the PV of a stream $X_1, X_2, ...$ and $PV_1$ be its PV at time $t = 1$ after payment of $X_1$, then

$$PV_0 = \text{Present Value}(X_1, PV_1).$$

5.2 Pricing in Markets for Dated Riskfree Cash Flows

This set of prices $P_t$ today of receiving one dollar at time $t$ in the future are important objects in finance. They are typically estimated from actual prices in financial markets.

Since most people are impatient, and would put more value today on receiving a dollar tomorrow than one year from now, you would expect the following property to hold:

$$P_1 > P_2 > P_3 > ...$$

If $P_1 < 1$ this can also be shown to be an implication of the no free lunch assumption, which is left as an exercise.

5.3 Interest Rates

Calculating present values is thus very simple as long as one know the prices $P_1, P_2, ...$. However, it is a long standing convention in finance to use interest rates instead of such prices. For each price $P_t$ there is a corresponding interest rate $r_t$. We will therefore need to spend some time on transformations involving interest rates.

Generally, the rate of return on an asset is

$$\text{Rate of return} = \frac{\text{Payments during period} + \text{Value at end of period}}{\text{Value at beginning of period}} - 1$$
Example
The simplest example of interest rates you find when you put money in the bank. The bank offers you a given interest on the balance of your account. If you put $100 in your bank account, and one year later take out all the money in the account, which by then includes interest of $8, the interest rate, or rate of return, on your bank account, is

\[
\text{rate of return = interest rate} = \frac{100 + 8}{100} - 1 = 0.08 = 8\%
\]

The relation between the prices \(P_t\) and interest rates are found as answers to the following question: How much would you have to invest now at the (per period) interest rate \(r_t\) to get one dollar at time \(t\)?

\[P_t(1 + r_t)^t = 1\]
which implies

\[P_t = \frac{1}{(1 + r_t)^t}\]  \hspace{1cm} (5.1)

and

\[r_t = t \left( \frac{1}{P_t} \right) - 1 = (P_t)^{-\frac{1}{t}} - 1.\]  \hspace{1cm} (5.2)

This type of interest is called discretely compounded interest. To complicate matters, there is another facet of interest rates, the frequency of compounding. We will return to this, for now we will use discrete compounding.

One thing to note about the expressions (5.1) and (5.2) for transforming between interest rates \(r_t\) and prices \(P_t\) is that they are one to one. If you know interest rates you also know prices, and vice versa. For most purposes, such as calculating present values, it is the prices that are of interest, not returns. (Ever seen a grocery store that quotes the prices of its apples as the \(-t^{th}\) root of its dollar price?) There are however also cases where interest rates may be more meaningful. The interest rate \(r_t\) denotes the percentage return on investing one dollar in a security that promises one dollar at time \(t\). The return is normalized to percent per period, so returns on securities with different maturity (different \(t^{'}s\)) can be compared. Many people will also compare investment opportunities by calculating an implied return on the project. Such comparisons are however full of pitfalls, many mistakes continue to be made by people who only use interest rates to compare investment opportunities.
5.4 Term Structure of Interest Rates

The plot of spot interest rates \( r_t \) against maturity \( t \) is called the *term structure of interest rates*. The term structure can take a multitude of shapes. Typically, it is rising, but it can also be decreasing, or even “humped.” Figure 5.1 shows an example term structure.

![Figure 5.1 Example Term Structure of Interest Rates](image)

The prices \( P_t \) (and, hence \( r_t \)) are usually estimated from prices of government fixed income securities, such as US Treasury bills and US Treasury bonds.

**Example**

A two-year Treasury bond with a face value of 1000 and an annual coupon payment of 8% sells for 982.50. A one-year T bill, with a face value of 100, and no coupons, sells for 90. Given these market prices, we can find \( P_1 \) and \( P_2 \) that gives the securities the correct prices:

\[
\begin{align*}
982.50 &= P_1 \times 80 + P_2 \times 1080 \\
90 &= P_1 \times 100
\end{align*}
\]

Solving these we find prices

\[
\begin{bmatrix}
P_1 = 0.9 \\
0.84
\end{bmatrix}
\]

and interest rates

\[
\begin{bmatrix}
r_1 = 11\% \\
r_2 = 9\%
\end{bmatrix}
\]
5.5 Net Present Value

The Present Value (PV) discussed above is the cost to obtain a set of future cash flows in the market. The Net Present Value (NPV) of an investment project is the difference between the Present Value and how much it costs you to generate the same cash flows (with your project). If the Net Present Value of a project is positive, it is obviously a valuable project: you are creating value; you generate a stream of cash flows at a cost lower than the market.

Example

An investment project promises the following future cashflows

\[
\begin{array}{ccc}
  t & 1 & 2 & 3 \\
  X_t & 1100 & 1100 & 1100 \\
\end{array}
\]

Starting the project costs 3000 today. If the interest rates are \( r_1 = r_2 = r_3 = 15\% \), is this a valuable project?

We calculate the NPV of the project by first finding the Present Value of the future cash flows as

\[
PV = \frac{1100}{(1 + 0.15)^1} + \frac{1100}{(1 + 0.15)^2} + \frac{1100}{(1 + 0.15)^3} = 2511.6
\]

Subtracting the cost today gives the NPV of the project:

\[
NPV = 2511.6 - 3000 = -488.4
\]

Since the \( NPV < 0 \), this is clearly a undesirable project.

5.6 Capital Budgeting

We use the term *Capital Budgeting* for the valuation and management of investment projects. This goes for any kind of investment project. The basic decision rule is to

**Invest in any project with a positive Net Present Value**

Positive NPV occurs when your cost to generate a stream of cash flows is less than the price that the market charges.

Example

Suppose you know how to generate a particular risky stream of cash flows by building new computer chips and selling them. The investment is cheap. You would have to pay a lot more to get the a stream of cash flows with the same properties (distribution over time and type of randomness) by combining equity, futures, bonds, etc.

Positive NPV reflects the presence of *economic rents*. (You own or know something that nobody else does.)
5.7 Perpetuities

Interest rates are useful for another reason, they can be used to simplify certain calculations. One example is the calculation of a perpetuity. A perpetuity is a sequence of payments each period into indefinite future. Its value is calculated as

\[ PV = \sum_{t=1}^{\infty} P_t X_t = \sum_{t=1}^{\infty} \frac{X_t}{(1 + r)^t} \]

In the case where the future cash flows are the same each year \( (X_t = X) \) and the interest rate \( r_t \) is constant, the above formula simplifies to\(^1\)

\[ PV = \frac{X}{r} \]

**Example**

What is the present value of an annual payment of $10 if the interest rate is 10%?

\[ PV = \frac{10}{0.1} = $100. \]

A growing perpetuity is a perpetuity that grows at a constant rate \( g \). The cash flow in year \( t \) is

\[ X_t = X_1 (1 + g)^{t-1} \]

Using this in the present value formula with constant interest rate \( r \),

\[ PV = \sum_{t=1}^{\infty} \frac{X_1 (1 + g)^{t-1}}{(1 + r)^t} \]

we can show that this simplifies to

\[ PV = \frac{X_1}{r - g} \]

\(^1\)This is shown relatively simply:

\[ PV = \sum_{t=1}^{\infty} \frac{X}{(1 + r)^t} = X \sum_{t=1}^{\infty} \left( \frac{1}{1 + r} \right)^t \]

\[ PV - PV \left( \frac{1}{1 + r} \right) \] is the price of receiving \( X \) at \( t = 1 \) only, i.e.

\[ PV - PV \left( \frac{1}{1 + r} \right) = PV \left( 1 - \frac{1}{1 + r} \right) = X \left( \frac{1}{1 + r} \right) \]

\[ PV = \frac{X}{r} \]
Example
Your bank offers you the following set of future cash flows: You receive 10 next year. Each subsequent year your payment will be 5% larger than the previous year. The interest rate $r$ is 10%. How much are you willing to pay the bank today for this set of cash flows?

$$PV = \sum_{t=1}^{\infty} \frac{10(1 + 0.05)^{(t-1)}}{(1 + 0.1)^t} = \frac{X_1}{r - g} = \frac{10}{0.1 - 0.05} = 20.$$ 

5.8 Annuities

An annuity is an asset that pays a fixed amount each year for a specified finite number of years. The present value of an annuity that last $T$ periods is found as:

$$PV = \sum_{t=1}^{T} \frac{X}{(1 + r)^t} = X \left[ \frac{1}{r} - \frac{1}{r (1 + r)^T} \right]$$

The term in brackets is called an annuity factor. Such annuity factors are tabulated in a lot of places. Most financial calculators will also provide them.

Example
You have just won the Lotto and can choose between $12$ million immediately or $1$ million per year for the next 20 years, payments starting one year from now. You can currently earn 5% by investing in treasury securities.

The choice should be based on the alternative with the highest present value. The present value of the latter alternative is calculated as

$$PV = 1\text{ mill} \left[ \frac{1}{r} - \frac{1}{r (1 + r)^T} \right]$$

$$= 1\text{ mill} \left[ \frac{1}{0.05} - \frac{1}{0.05(1 + 0.05)^{20}} \right]$$

$$= 12.46\text{ mill.}$$

The annual payment of 1 million is preferred to getting the 12 million immediately.

---

2 This is easily found as the difference between two perpetuities: one that starts one period from now, worth $\frac{X}{r}$, less the present value of one that starts $T + 1$ periods from now, worth $\frac{X}{r (1 + r)^T}$.
5.9 Compound Interest

Compounding refers to the frequency with which interest is added to the principal. To calculate the future value at time $t$ of compounding $n$ times per period at a constant interest rate $r$ we use the formula

$$FV_t = PV \left(1 + \frac{r}{n}\right)^{nt}$$

**Example**

If you invest $100 at a 10% annual interest rate, this will grow to

$$100(1 + 0.1)^{10} = 259.4$$

after 10 year with annual compounding, but it will grow to

$$100 \left(1 + \frac{0.1}{360}\right)^{360 \cdot 10} = 271.8$$

with daily compounding.

In the case of *continuous compounding* the above formula collapses to

$$FV_t = PV(e^{rt})$$

In continuous compounding, the return is computed as if you continuously received a dividend which you immediately reinvested. In discrete compounding, you receive a dividend only once (or a number of times $n$, depending on the case) every period. The continuous-time case is really the limit of discrete compounding, whereby the rate $r$ is paid and reinvested faster and faster:

$$\lim_{n \to \infty} \left( \left(1 + \frac{r}{n}\right)^n \right)^t = e^{rt}.$$

**Example**

If you invest $100 at a 10% annual interest rate, with continuous compounding this will grow to $100e^{0.1 \cdot 10} = 271.8$ after 10 years.

We can also find the present values for respectively discrete and continuous compounding as:

$$PV = FV_t \left(1 + \frac{r}{n}\right)^{-nt}$$

$$PV = FV_t(e^{-rt}),$$

If we are given prices $P_t$, we can find the corresponding interest rate with continuous compounding as follows

$$r_t = \frac{-\ln(P_t)}{t}$$
Example
If $P_t = 0.9$, the corresponding interest rate with continuous compounding is

$$r_t = \frac{-\ln(P_t)}{t} = -\ln 0.9 = 0.1053 = 10.53\%$$

With annual compounding the interest rate would have been

$$r_t = \frac{1}{P_t} = \frac{1}{0.9} = 0.1111 = 11.11\%$$

---

5.10 Valuing Fixed Income Securities

A fixed income security is a security that offers a predetermined sequence of future payments. The typical fixed income security is a bond.

Example
A US Government Bond (T Bond) with maturity 10 years and stated interest 7% is a promise to pay interest of 3.5% of the principal twice a year for 10 years, and repay the principal after 10 years.

Valuing bonds should by now be straightforward. We need to find the present value of the promised sequence of payments, using either prices $P_t$ or interest rates $r_t$.

Example
A bond promises the following sequence of payments:

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_t$</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>110</td>
</tr>
</tbody>
</table>

The interest rates $r_t$ and prices $P_t$ of future risk free cash flows are as follows

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_t$</td>
<td>5.3%</td>
<td>5.4%</td>
<td>5.6%</td>
<td>5.7%</td>
</tr>
<tr>
<td>$P_t$</td>
<td>0.95</td>
<td>0.9</td>
<td>0.85</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Bond Price = $\sum_{t=1}^{4} P_t X_t = 0.95 \cdot 10 + 0.9 \cdot 10 + 0.85 \cdot 10 + 0.8 \cdot 110 = 115$
5.11 Valuing Equities

An owner of one common stock owns a small part of the listed company the stock is issued by. Collectively, the equity owners are the residual claimants on the value of the firm, net of all the firm's liabilities. For one stock, however, this is not the relevant starting point for valuation. What counts is the cash flows accruing to the stock. For the company the cash flows are the dividends paid. The individual owner of a stock has an alternative source of cash flow, though: he can sell the stock to somebody else.

Example
You currently own one stock in the XYZ company. XYZ will pay dividend one year from now of one dollar. You also know that the price of one XYZ share one year from now, just after the dividend payments, will be 100 for sure. The current one year interest rate is 10%.

What is the current value of the stock?
This is just the present value:

\[ PV = \frac{1}{1 + r} (1 + 100) = 91.81 \]

But why is it possible to sell the share one year from now? Clearly the buyer of the share must believe that the value of one XYZ share at that time is 100. The source of the value must be cash flow from the XYZ share at some point in the future.

We are in other words in the following situation:

The price at time 1 is a present value of all the future dividends.

To value a stock it is not necessary to estimate some future stock price. One can also concentrate on the cash flows from the company, namely the dividends. The price of a stock is the present value of all future dividends.

\[ \text{Stock Price} = \sum_{t=1}^{\infty} P_t \text{Dividend}_t \]
But this just replaces one estimation problem with another. How to estimate all these future dividends? In general this is clearly impossible. However, if we assume that dividends will grow by a fixed percentage $g$ each year, this is an example of a growing annuity, which we have just seen, and if we further use a fixed interest rate $r$ we can calculate the price today as

$$\text{Stock Price} = \frac{\text{Dividend}_1}{r - g}$$

Example
The XYZ corporation will pay dividends of 1 next year. Dividends are expected to grow by 2% per year. The current interest rate is 10%. You estimate today's price of XYZ stock as

$$\text{Stock Price} = \frac{1}{0.1 - 0.02} = 12.5$$

5.12 Risky Cash flows

It's not obvious how you would "count" the number of units of dated cash flow if it is risky. Usually one unit equals one dollar of expected cash flow. That is, if $X_t$ now denotes the cash flow itself (a random variable), and $E[X_t]$ its expected value, then one measures the risky cash flow in terms of number of expected dollars, and writes:

$$PV = \sum_{t=1}^{T} P_t E[X_t].$$

But it is far from clear that you would want to use the same prices $P_t$ as when you were calculating the present value of future riskless cash flows. In fact, you will want to use different prices depending on the level of risk. We will be returning to this risky case.

References

Textbook References

Any basic text book on corporate finance, such as Brealey and Myers (2002) or Ross, Westerfield, and Jaffe (2005) covers this material in much more detail.
Problems

5.1 Present Value [3]
You are given the following prices $P_t$ today for receiving risk free payments $t$ periods from now.

\[
\begin{array}{ccc}
  t & = & 1 & 2 & 3 \\
  P_t & = & 0.95 & 0.9 & 0.85 \\
\end{array}
\]

1. Calculate the implied interest rates and graph the term structure of interest rates.

2. Calculate the present value of the following cash flows:

\[
\begin{array}{ccc}
  t & = & 1 & 2 & 3 \\
  X_t & = & 100 & 100 & 100 \\
\end{array}
\]

5.2 Borrowing [2]
BankTwo is offering personal loans at 10%, compounded quarterly. BankThree is offering personal loans at 10.5%, compounded annually. Which is the better offer?

5.3 Arbitrage [4]
You are given the following prices $P_t$ today for receiving risk free payments $t$ periods from now.

\[
\begin{array}{ccc}
  t & = & 1 & 2 & 3 \\
  P_t & = & 0.95 & 0.9 & 0.95 \\
\end{array}
\]

There are traded securities that offer $1 at any future date, available at these prices. How would you make a lot of money?

5.4 Bank Loans [2]
Your company is in need of financing of environmental investments. Three banks have offered loans. The first bank offers 4.5% interest, with biannual compounding. The second bank offers 4.3% interest, with monthly compounding. The third bank offers 4.25% with annual compounding. Determine which is the best offer.

5.5 Stock [4]
A stock has just paid a dividend of 10. Dividends are expected to grow with 10% a year for the next 2 years. After that the company is expecting a constant growth of 2% a year. The required return on the stock is 10%. Determine today's stock price.
5.6 Bonds [6]

You observe the following three bonds:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Price</th>
<th>Cashflow in period</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>95</td>
<td>100 0 0</td>
</tr>
<tr>
<td>B</td>
<td>90</td>
<td>10 110 0</td>
</tr>
<tr>
<td>C</td>
<td>85</td>
<td>10 10 110</td>
</tr>
</tbody>
</table>

1. What is the current value of receiving one dollar at time 3?

Consider now the bond D, with the following characteristics:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Cashflow in period</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>20 20 520</td>
</tr>
</tbody>
</table>

2. What is the current price of bond D?

Consider next bond E, which lasts for four periods. Bond E has the following characteristics:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Cashflow in period</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>10 10 10 110</td>
</tr>
</tbody>
</table>

3. If the market does not allow any free lunches (arbitrage), what is the maximal price that bond E can have?

5.7 Growing Perpetuity [8]

The present value of a perpetuity that pays $X_1$ the first year and then grows at a rate $g$ each year is:

$$PV = \sum_{t=1}^{\infty} \frac{X_1(1+g)^{t-1}}{(1+r)^t}$$

Show that this simplifies to

$$PV = \frac{X_1}{r-g}$$

5.8 Annuity [6]

Show that the present value of an annuity paying $X$ per period for $T$ years when the interest rate is $r$ can be simplified as

$$PV = \sum_{t=1}^{T} \frac{X}{(1+r)^t} = X \left[ \frac{1}{r} - \frac{1}{(1+r)^T} \right]$$
5.9 Stock [2]
The current price for a stock is 50. The company is paying a dividend of 5 next period. Dividend is expected to grow by 5% annually. The relevant interest rate is 14%. In an efficient market, can these numbers be sustained?

5.10 Growing Annuity [6]
Consider an T-period annuity that pays $X$ next period. After that, the payments grows at a rate of $g$ per year for the next $T$ years.
The present value of the annuity is

$$ PV = \sum_{t=1}^{T} \frac{X(1+g)^{(t-1)}}{(1+r)^t} $$

Can you find a simplified expression for this present value?

5.11 Jane [3]
Jane, a freshman in college, needs 55000 in 4 years to start studying for an MBA. Her investments earn 5% interest per year.

1. How much must she invest today to have that amount at graduation?

2. If she invested once a year for four years beginning today until the end of the 4 years how much must she invest?

5.12 Bonds [3]
The current interest rate is 7%. Given the opportunity to invest in one of the three bonds listed below, which would you buy? Sell short?

<table>
<thead>
<tr>
<th>Bond</th>
<th>Face value</th>
<th>Annual coupon rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1000</td>
<td>4%</td>
<td>1 year</td>
<td>990</td>
</tr>
<tr>
<td>B</td>
<td>1000</td>
<td>7.5%</td>
<td>17 years</td>
<td>990</td>
</tr>
<tr>
<td>C</td>
<td>1000</td>
<td>8.5%</td>
<td>25 years</td>
<td>990</td>
</tr>
</tbody>
</table>
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Chapter 6

Capital Budgeting

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6.1 Capital Budgeting

In this chapter we will look more specifically at the capital budgeting process. We will first look at some problems that tend to appear when calculating net
present values, such as the mistaking of accounting numbers for cash flows, how to treat sunk cost, repeated projects.

We will also discuss a number of alternative approaches that have been suggested to evaluate projects. While the most important is clearly NPV, it is necessary to be aware of the alternatives, and how they can be (and are) misused.

Note that the whole discussion in this chapter is in terms of risk free cash flows. This is for simplicity. As will become clear in later chapters, all what we are doing is also relevant for risky cashflows. It is merely a matter of adjusting prices, or discount factors.

6.2 Evaluating Projects Using NPV

Let us first give a reminder of how one calculates the NPV of a project.

6.2.1 Calculation of NPV

The NPV of a project is calculated by finding the Present Value of the future cash flows from the project and subtracting the current investment necessary to start the project. To find the present value we need either prices $P_t$ for cash flows at a future date $t$, or alternatively, an interest rate $r_t$ for cash flows at $t$. In capital budgeting, the interest rates implicit in prices of future cash flows are referred to as discount rates. Discount rates reduce the cash flow towards their present value.

The Present Value of a cash flow $X_t$ at $t$ is calculated as one of

$$PV = P_t X_t = \frac{1}{(1 + r_t)^t} X_t$$

**Example**

A project costs 100 today and has future cash flows

<table>
<thead>
<tr>
<th>$t$</th>
<th>$C_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
</tr>
</tbody>
</table>

If the interest rate $r_t = 5\%$ for $t = 1$ and $t = 2$, the PV is calculated as

$$NPV = \frac{50}{(1 + 0.05)^1} + \frac{80}{(1 + 0.05)^2} - 100 = 120.18 - 100 = 20.18$$

Since the NPV of the project is positive, this is a valuable project.
6.2.2 Only Cash Flows

The hard part of using the NPV rule in capital budgeting is in finding the relevant cash flow coming from the project. One should be very careful here. Only cash flows (actual dollar in/outflows) matter when computing NPV!

6.2.3 Accountants

Accounting numbers (profit, loss, book value) are absolutely irrelevant, except when they change tax payments. With the risk of angering accountants, keep the following in mind: “Accountants can be dangerous to your company's financial health. Only to be used to reduce tax payments.” Accountants have their own rules on determining (book) values and earnings. These are different from the ones used in finance. In particular, finance obtains values from market prices and the axioms of chapter 2. Earnings numbers would be irrelevant, were it not that their calculation determine taxes, and taxes are cash flows and therefore affect value.

6.2.4 After Tax Cash Flows

There is an important way in which accountants affect the NPV of a project, namely through depreciation. Normally, taxes are paid on the accounting profits from an investment. You can reduce these by depreciating the investment. This is called the depreciation tax shield. Note though that if you have no profits, you cannot use the depreciation to reduce your taxes. Depreciation can be used to reduce taxes only to the extent that you have positive profits.

6.2.5 What are Relevant Cash Flows?

Cash outflows (costs) that occurred in the past are sunk costs and are irrelevant for the computation of NPV. The market does not reward stupid past investments, so they do not enter in the computation of value. The fact that you already sunk so many billion dollars in the super conductor-collider does not make the project more attractive.

There are some relevant cash flows that are often forgotten, namely opportunity costs. When you make an investment, you may have to forgo money that you would otherwise have made automatically.

Example
The opportunity cost of getting a college education is the money you could have made flipping burgers at MacDonald's. This opportunity cost should be added in as cash outflow in the computation of the NPV of a college education.
Alternatively, opportunity costs should be acknowledged as cash flows in a separate project whose NPV has to be computed as well.

Example
In the college education case, think of it as two possible projects:

1. College education.
2. Working at MacDonald’s.

Compute the NPV of both projects and compare.

6.2.6 Inflation

Some care should be taken in the treatment of inflation. Most projects have some cashflows that will be adjusted as a result of inflation. Sales is a typical example of this, the prices you can sell goods for is easily adjustable. There are however some cashflows that are fixed in nominal terms, such as debt payments. Conceptually dealing with inflation should be straightforward. All that is needed is consistency. When using nominal cash flows, discount using the interest rates that apply to cash flows which are expressed in nominal terms. When adjusting cash flows for inflation (“constant dollars”), use “real” interest rates. People often compute the latter as the nominal rate less expected inflation. That is not necessarily correct, so maybe you want to forget about real cash flows entirely. Besides, economists don’t really agree on what inflation is.

Example
A project has the following projected cash flows in real terms.

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-100</td>
</tr>
<tr>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
</tr>
</tbody>
</table>

Inflation is 5% per year. The nominal discount rate is 10%.

To compute the NPV of the project, we can either use nominal cash flows and discount rate, in which case the calculation is

$$ NPV = \frac{50(1.05)}{1.1} + \frac{50(1.05)^2}{1.1^2} + \frac{40(1.05)^3}{1.1^3} - 100 = 28.1 $$

Alternatively we can use the real cash flows and instead adjust the discount rate to be a real one, which is approximated as

$$ \text{Real discount rate} \approx 10\% - 5\% = 5\% $$
to find the NPV of the project

\[
NPV = \frac{50}{1.05} + \frac{50}{1.05^2} + \frac{40}{1.0476^3} - 100 = 27.6
\]

which, except some roundoff error, is the same result as above.

### 6.2.7 Projects with Different Life Lengths

In case the projects have different life lengths, some care is necessary if you have to choose between them. If these projects are one shot projects, they should be compared using NPV. But when the projects are to be repeated (e.g. when a company is buying machines for its production), the projects must be made comparable in some way. One way of doing this is to find the periodicity of matching cycles and compute cash flows over this period.

**Example**

Consider two projects with life lengths 2 and 3 years. Make them into a comparable project by repeating the first one 3 times and the second one 2 times. This way you will have two projects with a 6 year life length to compare.

An equivalent procedure is called the *Equivalent Annual Cost Method*, and is discussed at length in standard textbooks.

### 6.3 Alternative Valuation Methods

A number of alternatives to NPV has presented itself. These range from methods that will agree with NPV "most of the time" to methods that will only agree with NPV by accident. While we advocate the use of NPV to make all decisions, it is sometimes useful to understand the alternatives and when they will lead to "wrong" decisions.

#### 6.3.1 Payback Period

The Payback Period of an investment project is defined as the number of years before a project returns its cost. The decision rule involving payback is to accept projects with a payback period shorter than some given period of time.

**Example**

A company is considering two projects with the following cash flows. The company is using a 15% discount rate on its investments.

<table>
<thead>
<tr>
<th>Project</th>
<th>Cost</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>100</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>50</td>
<td>50</td>
<td>100</td>
<td>300</td>
</tr>
</tbody>
</table>
The payback period for project 1 is 2 years, whereas the payback period for project 2 is 3 years. If the company only accepts projects with a payback less than two years, only project 1 will be acceptable. Project 1 is however the worst of these two project, it only returns its initial cost, and must therefore have a negative NPV. Let us calculate the NPV for the two projects:

\[
NPV_1 = -200 + \frac{100}{(1.15)^1} + \frac{100}{(1.15)^2} = -37.43
\]

\[
NPV_2 = -200 + \frac{50}{(1.15)^1} + \frac{50}{(1.15)^2} + \frac{100}{(1.15)^3} + \frac{300}{(1.15)^4} = 118.56
\]

Economically, project 2 is clearly the superior project, but it has a longer payback period. The example illustrates some of the weaknesses of using payback period as a decision criterion: There is no discounting of future payoffs. Payments beyond the payback period are ignored. There is no economic rationale for choosing a “cutoff” payback period.

There are attempts to “rescue” the payback period by discounting the future cashflows before calculating the payback period. While this would show that project 1 in the previous example is clearly undesirable, it will still not take account of cash flows beyond the (discounted) payback period.

### 6.3.2 Internal Rate of Return

The Internal Rate of Return (IRR) is defined as the interest rate that makes the NPV of a project zero.

**Example**
A project has cashflows

<table>
<thead>
<tr>
<th>Cost</th>
<th>X₁</th>
<th>X₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>150</td>
<td>150</td>
</tr>
</tbody>
</table>

The internal rate of return (IRR) of the project is the solution to

\[
NPV = \frac{150}{1 + IRR} + \frac{150}{(1 + IRR)^2} - 200 = 0
\]

In this case it is an easily solved quadratic equation

\[
IRR = 0.318726 \approx 31.9\%
\]
6.3 Alternative Valuation Methods

The decision rule is to accept projects where the IRR is higher than some given interest rate (hurdle rate, cost of capital). The IRR is a popular summary measure of the return from an investment. It is primarily useful because it is a relative measure, it is easy to compare the IRR of two investment projects. The IRR has a number of weaknesses, though, that should make one be very careful in using it.

One assumption one is making when using the IRR is that the interest rate is constant. As we have seen it is far from obvious that this is the case. What if the term structure of interest rates is not "flat"?

Example
Consider a project with the following cashflows

\[
\begin{array}{c|cccc}
  t & 0 & 1 & 2 & 3 \\
  \hline
  X_t & -100 & 10 & 10 & 110 \\
\end{array}
\]

The IRR in this case is 10%. The corresponding required interest rates are

\[
\begin{array}{c|ccc}
  t & 1 & 2 & 3 \\
  \hline
  r_t & 9\% & 10\% & 11\% \\
\end{array}
\]

You may be tempted to compare the average interest rate of 10% with the IRR in making your decision, but that would be a mistake. In this case it is necessary to calculate the NPV of the investment using the right values of \( r_t \)

\[
NPV = -100 + \frac{10}{(1 + 0.09)^1} + \frac{10}{(1 + 0.1)^2} + \frac{110}{(1 + 0.11)^3} = -2.13
\]

Another problem with IRR comes from the fact that it is a solution to a polynomial equation. First, it is hard to solve higher order equations, it must be done numerically. Also, most such equations have multiple solutions, some of which may be imaginary.

A useful tool, both for finding the relevant IRR, and understanding its properties, is the NPV Profile, which is a plot of the NPV as a function of the interest rate.

Example
A project has cashflows:

\[
\begin{array}{c|cc}
  t & 0 & 1 & 2 \\
  \hline
  X_t & -100 & 205 & -95 \\
\end{array}
\]

The next figure plots the NPV of the project for values of \( r \) from -40% to +40%
As the plot shows, there are two IRR’s for this project, which we can calculate to be approximately \(-29.2\%\) and \(34.2\%\). For all interest rates between these two the NPV of the project is positive.

The NPV profile can be used to illustrate another problem with IRR analysis. Sometimes you want the IRR on your project to be as low as possible!

Example
You are given two projects with cash flows

<table>
<thead>
<tr>
<th>Project</th>
<th>(t)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>-100</td>
<td>10</td>
<td>110</td>
<td>10%</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>100</td>
<td>-10</td>
<td>-110</td>
<td>10%</td>
</tr>
</tbody>
</table>

If your cost of capital is 9\% you may think you should accept both of these projects, but that would be a mistake. The second project is actually borrowing money at 10\%. When you borrow you want the lowest possible interest rate! If you look at the NPV profiles for these two projects, it is immediately obvious what the problem is:
6.3 Alternative Valuation Methods

The second project only has a positive NPV if the cost of capital is less than 10%!

As (yet another) area full of pitfalls in the application of IRR, the ranking of projects using IRR should be avoided, in particular when the projects are mutually exclusive.

Example
Consider the following two projects

<table>
<thead>
<tr>
<th>Project</th>
<th>t =</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>-100</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>23.4%</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>-1000</td>
<td>500</td>
<td>500</td>
<td>300</td>
<td>15.6%</td>
</tr>
</tbody>
</table>

The cost of capital is 10%. The projects are mutually exclusive. Which one to choose?

Based on IRR, project 1 seems like the better one. But this would be a mistake. If you calculate the net present values of the two projects, you find that \( NPV_1 = 24.3 \) and \( NPV_2 = 93.2 \). Project 2 is the one that adds most value, and it is the one that should be chosen.

The problem stems from the fact that the scale of the two projects are different. Project 1 earns a higher return, but only on a tenth of the investment of project 2. If you have available 1000 for your investment, what do you earn on the remaining 900 if you invest in project 1?

The internal rate of return is widely used in fixed-income analysis, where it is referred to as “bond yield,” or “yield to maturity,” but is very dangerous in that context.

Example
Two bonds with the same maturity and principal, but different coupons, can have different yields – this does not imply an arbitrage opportunity.

6.3.3 Profitability Index

The profitability index of a project is defined as the present value of the project divided by its cost. The decision rule used is to accept projects with a profitability index larger than one. This is equivalent to choosing projects with positive net present values. The rationale behind the use of the profitability index is that it is an attempt to get a relative measure of the desirability of a given project.

But the fact that it is a relative measure already points to a problem using the profitability index, it ignores scale, and we therefore have the same problems as we had using the IRR rule to rank mutually exclusive projects.
Example
Consider the same projects as above

<table>
<thead>
<tr>
<th>Project</th>
<th>t = 0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>IRR</th>
<th>PV</th>
<th>PI</th>
<th>NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-100</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>23.4%</td>
<td>124.3</td>
<td>1.24</td>
<td>24.3</td>
</tr>
<tr>
<td>2</td>
<td>-1000</td>
<td>500</td>
<td>500</td>
<td>300</td>
<td>15.6%</td>
<td>1093.1</td>
<td>1.1</td>
<td>93.1</td>
</tr>
</tbody>
</table>

Project 1 has the higher profitability index, but it has the lower NPV.

6.3.4 Accounting Measures of Return

The Average Accounting Return is defined as the per-year average accounting earnings after depreciation and taxes, divided by average book value. To use this measure for project valuation one would estimate the future accounting numbers for the project, calculate the accounting earnings for the future and compare the resulting estimate with some “hurdle” accounting rate of return. The problem with the accounting rate of return is that it uses accounting numbers, which usually have very little to do with cash flow. Accounting numbers are easily manipulated, and do not really reflect prices in a marketplace.

6.4 Some Fancy Acronyms

In recent years, a number of “fancy” acronyms have surfaced, often used by consultants to value companies. Despite their shortcomings, it’s important to understand these (and other new ideas), because shrewd managers and consultants abuse them.

Example
To increase bonuses, which often depend on return to equity, managers can take one-time charges for future costs. They can for example be justified as “re-organization costs.” When they occur, the actual cash outflows will not lower profit numbers, but the one-time charge lowers the (book) value of equity. As a result of the charge return on equity (accounting profit divided by book value of equity) increases. As will the managers’ bonuses...

6.4.1 EVA (Economic Value Added)

Comparison of operating profit with book value of debt and equity multiplied by “cost of capital.”

6.4.2 MVA (Market Value Added)

Market value of debt and equity less their book value.
6.4 Some Fancy Acronyms

6.4.3 TSR (Total Shareholder Return)

Return on equity, including dividends.

References

Any basic text book on corporate finance, such as Brealey and Myers (2002) or Ross et al. (2005), covers this material in much more detail.
Problems

6.1 *Projects* [2]
Two projects A and B have the following cashflows:

\[
\begin{array}{ccc}
X_0 & X_1 & X_2 \\
A & -4,000 & 2,500 & 3,000 \\
B & -2,000 & 1,200 & 1,500 \\
\end{array}
\]

1. Find the Payback periods for the two projects. Which project has the shortest payback period?

2. Calculate the Internal Rate of Return on the two projects. Which project has the higher IRR?

3. The discount rate is 10%. How would the NPV rule rank these two projects?

6.2 *Projects* [3]
A project costs 100 today. The project has positive cash flows of 100 in years one and two. At the end of the life of the project there are large environmental costs resulting in a negative cash flow in year 3 of −95. Determine the internal rate(s) of return for the project.

6.3 *Bonds* [4]
You are given the following information about three bonds.

<table>
<thead>
<tr>
<th>Bond</th>
<th>Year of Maturity</th>
<th>Coupon</th>
<th>Yield to Maturity</th>
<th>Bond Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>10%</td>
<td>7.5862%</td>
<td>1,043.29</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>20%</td>
<td>7.6746%</td>
<td>1,220.78</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>8%</td>
<td>9.7995%</td>
<td>995.09</td>
</tr>
</tbody>
</table>

Coupons are paid at the end of the year (including the year of maturity). All three bonds have a face value of 1,000 at maturity.

1. Find the time zero prices, \( P_1 \), \( P_2 \), and \( P_3 \), of one dollar to be delivered in years 1, 2, and 3, respectively.

2. Find the 1, 2, and 3 year spot rates of interest \( r_1 \), \( r_2 \) and \( r_3 \).

6.4 *Machine* [4]
A company is considering its options for a machine to use in production. At a cost of 47 they can make some small repairs on their current machine which will make it last
for 2 more years. At a higher cost of 90 they can make some more extensive repairs on their current machine which will make it last for 4 more years. A new machine costs 300 and will last for 8 years. The company is facing an interest rate of 10%. Determine the best action.

6.5 PI and NPV [2]
Show that a project with a positive NPV will always have a profitability index greater than 1.

6.6 Project [4]
A project has a cost of 240. It will have a life of 3 years. The cost will be depreciated straight-line to a zero salvage value, and is worth 40 at that time. Cash sales will be 200 per year and cash costs will run 100 per year. The firm will also need to invest 60 in working capital at year 0. The appropriate discount rate is 8%, and the corporate tax rate is 40%. What is the project’s NPV?

6.7 C&C [4]
The C&C company recently installed a new bottling machine. The machine’s initial cost is 2000, and can be depreciated on a straight-line basis to a zero salvage in 5 years. The machine’s per year fixed cost is 1500, and its variable cost is 0.50 per unit. The selling price per unit is 1.50. C&C’s tax rate is 34%, and it uses a 16% discount rate.

1. Calculate the machine’s accounting break-even point on the new machine (i.e., the production rate such that the accounting profits are zero).

2. Calculate the machine’s present value break-even point (i.e., the production rate such that the NPV is zero).

6.8 PillAdvent [6]
After extensive medical and marketing research, PillAdvent Inc, believes it can penetrate the pain reliever market. It can follow one of two strategies. The first is to manufacture a medication aimed at relieving headache pain. The second strategy is to make a pill designed to relieve headache and arthritis pain. Both products would be introduced at a price of 4 per package in real terms. The broader remedy would probably sell 10 million packages a year. This is twice the sales rate for the headache-only medication. Cash costs of production in the first year are expected to be 1.50 per package in real terms for the headache-only medication. Production costs are expected to be 1.70 in real terms for the more general pill. All prices and costs are expected to rise at the general inflation rate of 5%.
Either strategy would require further investment in plant. The headache-only pill could be produced using equipment that would cost 10.2 million, last three years, and
have no resale value. The machinery required to produce the broader remedy would cost 12 million and last three years. At this time the firm would be able to sell it for 1 million (in real terms). The production machinery would need to be replaced every three years, at constant real costs. For both projects the firm will use straight-line depreciation. The firm faces a corporate tax rate of 34%. The firm believes that the appropriate discount rate is 13%. Capital gains are taxed at the ordinary corporate tax rate of 34%.

What pain reliever should the firm produce?
Chapter 7

Valuation under Uncertainty: The CAPM

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7.1 Asset Pricing Theory

The purpose of Asset Pricing Theory is to understand the pricing of risk. That involves two things:

1. Recognizing risk categories, the ability to classify risky cash flows in “bins” of equal “risk.”

2. Pricing each risk category, finding a price per dollar of expected cash flow.
Valuation under Uncertainty: The CAPM

To find the price $PV$ today of a given future risky cash flow we need to use

$$PV = P_tE[X_t]$$

But note that $P_t$ is now the price of future risky cash flows. The intuitive implication of risk is that the higher the risk, the lower the current price for the future risky cash flows. It will turn out that our intuition is correct.

We will in this chapter mainly work with returns. Returns is the transformation of prices into equivalent interest rates, as discussed in chapter 5, defined by

$$P_t = \left( \frac{1}{1 + r_t} \right)^t$$

For the price to decrease when risk increases (move to a higher "risk category") it must be the case that the interest rate increases when risk increases. The "asset pricing question" is then to find what is the relevant measure of risk.

To summarize what we will be showing in this chapter: For a given asset, the asset's volatility is not an appropriate measure of risk. Rather, the relevant measure of risk depends on how much this particular asset contributes to a measure of overall risk. The central result of asset pricing theory we will show is therefore that

Only the covariance with some diversified "benchmark portfolio" is priced.

### 7.2 Portfolio Returns

We will go through the "classical" Capital Asset Pricing Model (CAPM) and see how risk is priced there. To do so, we must first cover some preliminaries on portfolios and returns. The intuition for the CAPM is based on per period returns for a portfolio of assets. An investor's preferences are supposed to be such that he prefers higher expected portfolio returns, but dislikes variability, measured by the variance, or equivalently standard deviation, of the portfolio returns.

Let $\bar{r}_i$ be the return on asset $i$. There is no indexing by time, for now we only deal with one period. The return is random, and $E[\bar{r}_i]$ is its expected value. $\sigma_i^2$ is the variance of returns on asset $i$. The standard deviation of returns is then $\sigma_i$. For two assets $i$ and $j$ with random returns, $\sigma_{ij}$ is their covariance. The correlation between the returns on the two assets is $\rho_{ij}$.

Let $\omega_i$ be the fraction of your portfolio invested in asset $i$. This is commonly called the weight of asset $i$ in the portfolio. The expected return and variance of a portfolio $p$ is

$$E[r_p] = \sum_i \omega_i E[\bar{r}_i]$$
\[ \sigma^2(\bar{r}_p) = \sum_i \sum_j \omega_i \omega_j \sigma_{ij} \]

**Example**
You can choose to invest in two shares, CALTEX and EXXOFF.

<table>
<thead>
<tr>
<th></th>
<th>( E[r] )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CALTEX</td>
<td>10% 10%</td>
<td></td>
</tr>
<tr>
<td>EXXOFF</td>
<td>15% 20%</td>
<td></td>
</tr>
</tbody>
</table>

The correlation between the returns on the two shares is 0.15.

Your portfolio consists of 100 CALTEX shares and 50 EXXOFF shares. The current price of CALTEX is 50, the current price of EXXOFF is 100.

The total value of your CALTEX shares is 100 \( \times \) 50 = 5000. The value of your EXOFF shares is 50 \( \times \) 100 = 5000. Your total wealth is the sum of these values, 10000. The fraction of your wealth invested in each share, or portfolio weights, is a half for each asset:

\[
\omega_1 = \omega_{\text{CALTEX}} = \frac{5000}{10000} = \frac{1}{2} \\
\omega_2 = \omega_{\text{EXXOFF}} = \frac{5000}{10000} = \frac{1}{2}
\]

The expected return and standard deviation of the portfolio is

\[
E[\bar{r}_p] = \omega_1 E[\bar{r}_1] + \omega_2 E[\bar{r}_2] = \frac{1}{2} \times 10\% + \frac{1}{2} \times 15\% = 12.5\%
\]

\[
\sigma^2(\bar{r}_p) = \omega_1^2 \sigma^2(\bar{r}_1) + 2\omega_1 \omega_2 \sigma(\bar{r}_1) \sigma(\bar{r}_2) \rho(\bar{r}_1, \bar{r}_2) + \omega_2^2 \sigma^2(\bar{r}_2) \\
= 0.5^2 \times 0.1^2 + 2 \times 0.5 \times 0.5 \times 0.1 \times 0.15 + 0.5^2 \times 0.2^2 = 0.014
\]

\[
\sigma(\bar{r}_p) = \sqrt{\sigma^2(\bar{r}_p)} = 11.8\%
\]

### 7.3 The Distinction between Diversifiable and Nondiversifiable Risk

Some useful intuition can be had at this point, namely that purely idiosyncratic risk (risk unique to an asset) should not be priced in equilibrium. The point is usually made the following way. From the set of possible assets, choose randomly a number \( n \) of the assets, such as stocks at the NYSE, and calculate the historical return and standard deviation of an equally weighted portfolio (One in which each asset has a weight \( \omega = \frac{1}{n} \)). Repeat this for increasing numbers of \( n \). A picture like figure 7.1 emerges.

With only one or two assets in the portfolio, the standard deviation is large. By increasing the number of assets the standard deviation decreases, but only up
to some level. This is the finance way of telling you the folly of putting all your eggs in one basket. Now ask yourself: Should one care about the diversifiable risk? Not if you aren’t rewarded for holding it. That is precisely what happens in the CAPM.

### 7.4 The Set of Efficient Portfolios

Suppose first that the investor can only invest in risky assets. What are the feasible portfolios for such an investor? The investor takes the set of assets, expected returns, variances and covariances as given. The only thing the investor can vary are the weights. By varying the weights the investor can generate a feasible set of portfolio combinations. Figure 7.2 shows an example of a feasible set.

If we accept that investors want high expected portfolio return but dislike portfolio volatility, clearly each investor’s optimal portfolio is one that maximizes portfolio expected return for a given level of portfolio standard deviation. The set of such portfolios is called the efficient set, and it is the upward sloping part of the curve in figure 7.2.
7.5 The Possibility Set with a Risk Free Security

The picture changes with the added possibility of investing in a risk free security. Let $r_f$ be the return on such a security. A typical example of a risk free security is a short term government (Treasury) bill. A risk free security has standard deviation of zero. If we combine a risky security with the risk free security, the combination maps as a line in the mean-standard deviation plot, as shown in figure 7.3, where we combine the risk free asset with assets $m$ and $p$. As is obvious from the figure, the only risky asset that it makes sense for the investor to use is $m$, the tangency portfolio. Assets on the line from $r_f$ through $m$ dominates all other feasible assets. With a risk free asset, the efficient set is therefore the line from $r_f$ through the tangency portfolio $m$.

7.6 The Capital Asset Pricing Model

We are now ready to discuss the central intuition in the Capital Asset Pricing Model (CAPM). The arguments above tells us that any investor will want to hold as his portfolio a combination of the risk free asset and the portfolio $m$ of risky asset. But then the total demand by investors of securities will also be some combination of the risk free asset and the portfolio $m$. In an economic equilibrium, demand has to equal supply. A logical consequence is then that the portfolio $m$ has to equal the total supply of risky assets, the market portfolio. To
Figure 7.3 The Feasible Set of Assets. Risk Free Asset combined with portfolios \( p \) and \( m \).

make demand equal supply, the prices of risky assets will have to adjust, changing expected returns and return variances. In the end, each investor's portfolio consists of a combination of the risk-free asset and the market portfolio \( m \) of all risky assets. Hence, in equilibrium, the market portfolio must be mean-variance efficient, which really means that

The market portfolio has the highest possible reward-to-risk ratio ("Sharpe ratio")

Mathematically, this is equivalent to:

\[
E[r_i] = r_f + \beta_i (E[r_m] - r_f)
\]  
(7.1)

where

\[
\beta_i = \frac{\text{Covariance}(r_i, r_m)}{\text{Variance}(r_m)}.
\]

\( \beta_i \) is asset \( i \)'s beta. From the CAPM equation, we see that the only thing that affects an asset's risk is the asset's covariance with the market portfolio.

Equation (7.1) very clearly indicates that average returns are only determined by risk, where risk is not measured as volatility, but as "covariance with the return on a benchmark portfolio" (in this case, the market portfolio).
7.7 Using CAPM for Pricing

Now we have arrived at a feasible way of pricing future risky cashflows. Recall we were to price risky cashflows as

\[ P = P_t E[X_t] = \left( \frac{1}{1 + r_t} \right) E[X_t] \]

The need was to find either the price \( P_t \) or the interest rate \( r_t \) to plug into the pricing relation. The CAPM relation (7.1) above provides exactly that, a way of estimating the required return \( r_t \) for a given risk class. To implement this in practice we need to estimate the beta, the covariance between the return on the cash flows and the market portfolio. If for example the asset in question is a stock, we can easily estimate this from historical data of stock returns. The only problem is the choice of the market portfolio. In practice we usually chose a broad based stock index as a representative market portfolio, and calculate beta from the covariance of these returns.\(^1\)

Example

The estimated covariance between stock J and the S&P 500 stock market index is 0.04. The variance of the S&P 500 index is 0.02. The beta of stock J is

\[ \beta_J = \frac{0.04}{0.02} = 2 \]

If the historical risk premium \( (r_{SP500,t} - r_f,t) \) is estimated as 6%, and the current risk free interest rate is 4%, we can estimate the expected return on stock J as

\[ E[r_J] = r_f + (E[r_m] - r_f)\beta_J = 0.04 + 0.06 \cdot 2 = 16\% \]

One remark, though. It turns out that the statement in the previous section, "Average returns are determined solely by covariation with returns on some benchmark portfolio," is vacuous without qualification about what that benchmark portfolio is (Roll, 1977). The CAPM identifies the market portfolio as the benchmark portfolio. But is it observable? If not, the CAPM may not have much empirical content. This point is very relevant for an extremely mundane practice, namely, mutual fund performance evaluation. Discussion has to be delegated to an investments text, though.

\(^1\) Alternatively Beta can be interpreted as the (OLS) slope coefficient in a projection of its return onto the return of the market.
7.8 Using CAPM for Capital Budgeting

So far in our discussion of capital budgeting we have not mentioned risk, all examples were ones with risk free cash flows. The CAPM gives us a tool to adjust for the riskiness of a project, one that is remarkably simple to apply. Risk is measured by the beta of a project, the covariance of the project returns and the market returns. Given the project beta, use the CAPM equation (7.1) to find the (risk adjusted) discount rate for the project. Use this discount rate to discount the expected cash flows from the project. Hence, to value a risky cash flow \( X_1 \) which accrues “tomorrow,” and which costs \( X_0 \), we first compute the project beta as

\[
\beta = \frac{\text{Covariance of } \frac{X_1}{X_0} \text{ and } r_m}{\text{Variance of } r_m},
\]

where \( r_m \) denotes the return on the market portfolio. We next use this “beta” to compute the required discount rate:

\[
r = r_f + \beta (E[r_m] - r_f),
\]

where \( r_f \) is the risk free rate. The Net Present Value (NPV) of the project is then calculated as

\[
\text{NPV} = \frac{E[X_1]}{1 + r} - X_0.
\]

Actually, the astute reader will have noticed that the above procedure isn’t quite correct, despite its widespread usage. Our principle of valuation has been to compare the cost \( X_0 \) of a project to the price \( P \) it takes to buy the same cash flow \( X_1 \) in the marketplace. The market computes the \( \beta \) of \( X_1 \), not with respect to \( X_0 \), but with respect to \( P \), as follows:

\[
\beta = \frac{\text{Covariance of } \frac{X_1}{P} \text{ and } r_m}{\text{Variance of } r_m}.
\]

This technical detail makes valuation with CAPM rather complicated. We will not elaborate here, because we would like to move on and introduce a far more powerful valuation procedure under uncertainty. The interested reader is referred to Ekern (2006).

**Example**

A project costs 100 today and has expected cashflows \( E[X_1] \) of 120 next period. The covariance between the project return and the market portfolio is 0.0015. The variance of the market portfolio is 0.001. The current risk free rate is 10%, and the expected return on the market portfolio is 12.5%. Determine the NPV of the project.
7.9 Empirical Evidence

We first find the relevant discount rate for the project, by finding the project beta

\[
\beta = \frac{\text{cov}(\text{project return}, r_m)}{\text{var}(r_m)} = \frac{0.0015}{0.001} = 1.5.
\]

Given the beta one calculates the discount rate as

\[
r = r_f + \beta(E[r_m] - r_f) = 0.1 + 1.5(0.125 - 0.1) = 13.75\%.
\]

Use this discount rate to discount the future cash flows:

\[
NPV = \frac{E[X_t]}{1 + r} - \text{costs} = \frac{120}{1 + 0.1375} - 100 \approx 105.50
\]

7.9 Empirical Evidence

The Efficient Markets Hypothesis (EMH) explains why there is little predictability in securities prices. The EMH implies that average returns are determined solely by risk. Asset pricing theory determines the content of "risk," namely covariance with the returns on a diversified benchmark portfolio. Historical data support this proposition across broad categories of assets. However, a closer look reveals ample violations. These violations can be "worked away" by a clever choice of benchmark portfolio, but that does not advance our understanding of financial markets. This is the state of the art...

The empirical evidence is only weak (at best). We want to reflect on this a little bit, because the CAPM is the main model on which actual corporate finance decisions (including litigation) are based. The model is theoretically compelling (beautiful logic), but that is not enough.

In fact, testing the CAPM on historical NYSE or NASDAQ data is like testing the proposition that \( g = 9.8 \text{m/s}^2 \) on naturally falling objects. There are lots of things that can go wrong. Besides picking the wrong "market portfolio," investor beliefs may be wrong, there may be sample selection bias, investors may care about more than volatility (for example downside risk, which manifests itself as skewness), there are taxes, the markets may not continuously be in equilibrium, etc.

7.10 Experimental Evidence

To gauge whether the CAPM describes prices in actual financial markets, an obvious alternative is to look at experimental evidence. Experimental financial markets were set up at Caltech in a very simple way. If the CAPM did not emerge in our simple setting, what hope would there be that it does in the far more
complex setting of the NYSE? The results of the experiment shows that, yes, the CAPM emerges, but *very slowly*. The "price discovery" process is painstakingly slow.

The conclusion to draw from the experimental evidence: the idea behind the CAPM (that financial markets equilibrate to the point that only covariation with some aggregate benchmark portfolio is priced) has scientific support, but it comes about so slowly that the intermediate "price discovery" may need to be studied more carefully.

### 7.11 The Arbitrage Pricing Theory

In the CAPM the only source of uncertainty is the covariance with the *market portfolio*. Many view this simple relationship as too simplistic, both on theoretical and empirical grounds. "There must be some other things influencing the stock market." The *Arbitrage Pricing Theory* (APT) is may be best viewed as a way of justifying running a multivariate regression on a given set of "factors" that one think may be relevant in pricing equities, such as risk free interest rates, inflation, state of the business cycle, the probability of corporate default, interest rates on corporate bonds, and so forth. To implement the APT one would need to choose a set of "factors," or observable market variables that one thinks influences stock returns. Given this one estimates the coefficients in a regression of stock returns on the "factors." These coefficients are then used in a similar way to the beta in the CAPM in setting the required return.

### 7.12 When the CAPM Would Fail

In the CAPM (and all generalizations), the view of "risk" is narrow:

\[
\text{Risk} = \text{portfolio volatility}
\]

This is at the heart of the central result that only "beta" is priced: a security returns more if it contributes more to the volatility of optimal portfolios, and this contribution is measured by "beta." But there may be other sources of risk. For the same volatility, a security with higher *skewness* may be perceived to be more risky. In fact, many financial securities have highly skewed payoffs. The prototype of such a security is the *option*. Many securities that are issued by corporations have option-like payoffs. The CAPM is not designed to price such securities. Hence, we will have to consider another approach. That will be done in Part III.
References

Original References

The link between number of assets in a portfolio and the portfolio variance is shown in Wagner and Lau (1971). The original sources on the CAPM is Sharpe (1963), Lintner (1965) and Mossin (1966). The Arbitrage Pricing Theory is primarily due to Ross (1976). The experiments mentioned are discussed in Bossaerts and Plott (2004). Current views on the CAPM are found in Perold (2004) and Fama and French (2004).

Textbook References

The material on the classical CAPM is typically covered in an investment text like Bodie et al. (2005) or Haugen (2001). A good proof of the CAPM is in Mossin (1973).
Problems

7.1 Portfolio [3]
You can choose to invest in two shares, A and B:

<table>
<thead>
<tr>
<th></th>
<th>$E[r]$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12.5%</td>
<td>15%</td>
</tr>
<tr>
<td>B</td>
<td>16%</td>
<td>20%</td>
</tr>
</tbody>
</table>

Plot, in a mean standard deviation diagram, expected return and standard deviation for portfolios with weights in A of 0, 0.25, 0.5, 0.75 and 1.0. Do this for correlations between the returns of A and B of $-1$, 0 and $+1$. What does this tell you about how diversification possibilities varies with covariances?

7.2 CAPM [2]
The current risk free interest rate is 5%. The expected return on the market portfolio is 14%. What is the expected return of a stock with a beta value of 0.5?

7.3 Beta [2]
Stock A has an expected rate of return of 15%. Todays expected return on the market portfolio is 10%. The current risk free interest rate is 7.5%. What is the beta of stock A?

7.4 Project [3]
A project with a beta of 1.5 has cash flows 100 in year 1, 200 in year 3, 500 in year 4 and 100 in year 6. The current expected market return is 10%. The risk free interest rate is 5%. What is the highest cost that makes this project worth investing in?

7.5 Portfolio [2]
Stock A has an expected return of 10% and a standard deviation of 5%. Stock B has an expected return of 15% and a standard deviation of 20%. The correlation between the two is shares is 0.25. You can invest risk free at a 5% interest rate. What is the standard deviation for a portfolio with weights 25% in A, 25% in B and 50% in the risk free asset?

7.6 Q [2]
Equity in the company Q has an expected return of 12%, a beta of 1.4 and a standard deviation of 20%. The current risk free interest is 10%. What is the current expected market return?
7.12 When the CAPM Would Fail

7.7 Line [4]
You can invest in two assets. One is a risk free asset yielding an interest rate of \( r_f \). The other is an asset with expected return \( E[r_1] \) and standard deviation \( \sigma_1 \). Show that combinations of these two assets map as a straight line in a mean-standard deviation plot.

7.8 1&2 [2]
A portfolio is made up of 125% of stock 1 and -25% of stock 2. Stock 1 has a standard deviation of 0.3, and stock 2 has a standard deviation of 0.05. The correlation between the stocks is -0.50. Calculate both the variance and the standard deviation of the portfolio.

7.9 A, B & C [4]
You are given the following information about three stocks

<table>
<thead>
<tr>
<th>Stock</th>
<th>Expected Return</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.06</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>C</td>
<td>0.2</td>
<td>0.375</td>
</tr>
</tbody>
</table>

The correlation between B and C is 0.2

1. Suppose you desire to invest in any one of the stocks listed above (singly). Can any be recommended?

2. Now suppose you diversify into two securities. Given all choices, can any portfolio be eliminated? Assume equal weights.

7.10 EotW [5]
Eyes of the World Corporation has traditionally employed a firm-wide discount rate for capital budgeting purposes. However, its two divisions, publishing and entertainment, have different degrees of risk given by \( \beta_P = 1 \) and \( \beta_E = 1.5 \), where \( \beta_P \) is the beta for publishing and \( \beta_E \) the beta for entertainment. The beta for the overall firm is 1.3. The risk free rate is 5% and the expected return on the market is 15%. The firm is considering the following capital expenditures:

<table>
<thead>
<tr>
<th>Publishing</th>
<th>Entertainment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed Project</td>
<td>Initial Investment</td>
</tr>
<tr>
<td>P1</td>
<td>1M</td>
</tr>
<tr>
<td>P2</td>
<td>3M</td>
</tr>
<tr>
<td>P3</td>
<td>2M</td>
</tr>
</tbody>
</table>
1. Which projects would it accept if it uses the opportunity cost of capital for the entire company?

2. Which projects would it accept if it estimates the cost of capital separately for each division?

7.11 Misui [4]

Misui, Inc is a levered firm with a debt-to-equity ratio of 0.25. The beta of common stock is 1.15, while that of debt is 0.3. The market premium (expected return in excess of the risk free rate) is 10% and the risk free rate is 6%. The CAPM holds.

1. If a new project has the same risk as the common stock of the firm, what discount rate should the firm use?

2. If a new project has the same risk as the entire firm (debt and equity), what discount rate should the firm use? (Hint: Betas are "additive." In the present context, this means the following. The value of the firm \( V \) equals the sum of the value of equity \( E \) and debt \( B \). That is, the firm is really a portfolio of equity and debt. As a result, the beta of the firm equals the weighted average of the betas of equity and debt, where the weights are given by the ratios of \( E \) over \( V \) and \( B \) over \( V \), respectively.)
Part III

Multiperiod Pricing and Derivatives
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Chapter 8

Valuing Risky Cash Flows

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8.1 Valuation of Dated, Risky Cash Flows

We now take a very different approach to valuing dated, risky cash flows. We have till now considered the following two strands of thought:

First we looked at the value of risk free cash flows at different dates. To find the current value of a stream of future riskfree cash flows \( X_1, X_2, \ldots, X_T \) at dates \( t = 1, 2, \ldots, T \) we used prices \( P_t \) of one dollar to be delivered at time \( t \). This price is related to the \( t \)-period interest rate \( r_t \) as follows:

\[
P_t = \frac{1}{(1 + r_t)^T}.
\]

The current value of the stream of future cash flows equals

\[
P_1 X_1 + P_2 X_2 + \ldots + P_T X_T,
\]

as if it were a basket of different units of each of \( T \) goods. The current value price is also called the Present Value (PV).
Secondly, we looked at the pricing of risky cash flows, using the prototypical asset pricing model, the CAPM. What is the price $P$ of a future, risky cash flow $X$ which equals $E[X]$ on average? To answer this, we need to first recognize what risk category $X$ belongs to. This means that we have to compute the $\beta$ ("beta") of the cash flow, defined to be:

$$\beta = \frac{\text{covariance}(X, r_m)}{\text{variance}(r_m)},$$

where $r_m$ denotes the return on the market portfolio.

If we were to compute the price $P_{\beta'}$ per unit of expected payoff for a cash flow with $\beta$ equal to $\beta'$, we would according to the CAPM calculate it as

$$P_{\beta'} = \frac{1}{1 + r'},$$

where

$$r = r_f + \beta'(E[r_m] - r_f)$$

Here $r_f$ denotes the riskfree rate and $E[r_m]$ the expected return on the market portfolio. To find the present value of the future cash flows we would calculate

$$PV = P_{\beta'} E[X].$$

We will now turn to a second way of evaluating future risky cash flows based on the possible future "states."

### 8.2 States

A state is an enumeration of what can happen in the future.

**Example**

There are two possible states of the economy next period: "boom" or "bust." In the boom ("up") state the economy grows by 25%. In the "bust" or "depression" ("down") state, the economy suffers from a 25% negative growth. We will use $u$ (up) and $d$ (down) to index the two states.

### 8.3 States and Digital Options

Consider a useful financial security called a digital option. A digital option is a security that pays one dollar if and only if a certain state occurs.
Example
In the example with two states, "boom" or "bust," there are two digital options traded. One pays $1 if the economy is booming, the other pays $1 if the economy is in the doldrums. The current price of one "boom" digital option ($u$) is $0.35, the current price of one "bust" digital option ($d$) is $0.45. As we see, it is less valuable to get a dollar when the economy is booming, and more valuable when it is doing badly.

Knowing the prices of digital options is very useful, because by value additivity they can be used to price any other security with known state-contingent cashflows. Prices of such digital options are often called state prices.

Example
In the boom/bust example, there are two securities outstanding. One is a risk free bond promising to pay $1 in either state. The other is a risky stock that pays $1.5 if the economy is booming and $0.5 in the other state. What is the current value of these two securities?

To price the risk free security we note that the payoffs from it ($1 in either state) is the same as the payoffs from a portfolio consisting of one "boom" digital option and one "bust" digital option. The price today of the risk free security has to equal

$$\text{price} = u \cdot 1 + d \cdot 1 = 0.35 + 0.45 = 0.8$$

This can be used to calculate the risk free interest rate

$$\frac{1}{1 + r} = 0.8 = u + d$$

$$r = \frac{1}{0.8} - 1 = 25\% = \frac{1}{u + d} - 1$$

Similarly, to price the risky stock we note that the cashflows next period from this security equals the payoff of a portfolio consisting of 1.5 "boom" digital options and 0.5 "bust" digital options. We can therefore price it as

$$\text{Price} = u \cdot 1.5 + d \cdot 0.5 = 0.35 \cdot 1.5 + 0.45 \cdot 0.5 = 0.75$$

As the example illustrates, we can find the price of a risk free security (and hence the risk free interest rate) by observing the prices of all possible digital options. The price of a risk free security is

$$P_f = \sum_s I_s,$$

where $s$ enumerates all possible states, and the risk free rate is

$$r = \frac{1}{P_f} - 1 = \frac{1}{\sum_s I_s} - 1.$$
Since the risk free interest rate usually is positive (time value of money), this implies that
\[ \sum s I^s < 1 \]

Example
In the boom/bust example, we know the prices of two digital options \( I^u = 0.35 \) and \( I^d = 0.45 \).
\[ P_f = 0.35 + 0.45 = 0.80 \]
The risk free interest rate is
\[ r = \frac{1}{P_f} = \frac{1}{0.80} = 25\% \]

8.4 Expectations and Digital Options

As we see, the prices of digital options are useful for valuing future risky cashflows. But for convenience we will often use an equivalent formulation of the valuation problem.

Example
In the boom/bust example, we know the prices of two digital options \( I^u = 0.35 \) and \( I^d = 0.45 \). To value a security that has cashflows \( X^u = 10 \) and \( X^d = 5 \) we calculate
\[ \text{Value} = I^d X^d + I^u X^u = 10 \cdot 0.45 + 5 \cdot 0.35 = 6.25 \]
This can be thought of as a portfolio of 10 "up" digital options and 5 "down" digital options.
An equivalent way of valuing the security is to find the value at time 1 of this portfolio and then discounting it at the risk free rate. The value at time 1 of an investment of \( X \) at time 0 is \( X(1 + r) \). If we define
\[ p^u = I^u (1 + r) \]
and
\[ p^d = I^d (1 + r), \]
the time 1 value of owning one digital option today, the valuation of the security is done by multiplying this value with the number of each digital option we have, and then discounting at the risk free rate.
\[ \text{Value} = \frac{1}{1 + r} \left( p^d X^d + p^u X^u \right) \]
The numbers $p^u$ and $p^d$ are calculated as:

$$p^u = 0.35 \cdot 1.25 = 0.4375$$

$$p^d = 0.45 \cdot 1.25 = 0.5625$$

and we find the current value of the security as

$$\text{Value} = \frac{1}{1 + r} \left( p^d X^d + p^u X^u \right) = \frac{1}{1 + 0.25} \left( 0.4375 \cdot 10 + 0.5625 \cdot 5 \right) = 6.25$$

Note here the fact that the sum of $p^u$ and $p^d$ is $0.4375 + 0.5625 = 1$

We can calculate the current value of a security with time 1 state contingent payoffs $X^s$ as

$$\text{value} = \frac{1}{1 + r} \sum_s p^s X^s \quad (8.1)$$

It is actually a general property of the calculated $p^s$ that each is less than one and they sum to one. But then they act like probabilities.

At this point it may be useful to recall from basic probability theory the definition of a probability. Loosely stated, these are the following. A probability measure has three defining properties. Given a set of outcomes, a number between 0 and 1 represents the probability of an outcome. The probabilities for all possible (mutually exclusive) outcomes sum to one. If two outcomes are mutually exclusive, the probability of the joint event is the sum of the probabilities of the two individual events.

We leave it to the reader as a homework problem to show that the numbers $p^s$ satisfy these conditions, and therefore is a probability measure. The equation (8.1) can thus be thought of as discounting an expected value using the $p^s$ as probabilities:

$$\text{Value} = \frac{1}{1 + r} \sum_s p^s X^s = \frac{1}{1 + r} E^*[X]$$

We will use the term state price probability for the numbers $p^s$. This expression here has earned the name risk neutral valuation in finance, and the numbers $p^s$ are often called risk neutral probabilities. But some care is advised here. These names are misnomers. The numbers $p^s$ are numbers calculated on the basis of market prices of digital options, they do not equal the true probabilities for the states $s$ occurring. Also, nowhere do we say anything about risk aversion. In particular, we don't claim that the market is risk neutral.
8.5 Trees

The payoff pattern in this approach is often represented graphically in terms of a tree:

\[ P \xrightarrow{} X^1 \xrightarrow{} X^2 \xrightarrow{} X^3 \]

Most of the time we will limit ourselves to two states per period. This is called a binomial tree.

8.6 Summarizing

We have shown that we can value risky, dated cashflows the following way: Determine the payoffs \( X^s \) in all future states \( s \). Find the state price probabilities \( p^s \). Calculate today's value as

\[
\text{value} = \frac{1}{1+r} \sum_s p^s X^s = \frac{1}{1+r} E^s[X^s]
\]

where \( r \) is the risk free interest rate. In many cases we will limit ourselves to two states, and calculate

\[
\text{value} = \frac{1}{1+r} \left( p^u X^u + p^d X^d \right) = \frac{1}{1+r} \left( p^u X^u + (1-p^u)X^d \right)
\]

References

The notion of state prices goes back to Debreu (1959) and Arrow (1964). The notion of risk neutral valuation was first introduced in Cox and Ross (1976). A reference on the use of digital options for pricing is Ingersoll (2000). Varian (1992) discusses the state price approach. A more formal definition of probability can be found in any basic textbook in probability theory, such as Ross (1993).
Problems

8.1 States [4]
Two possible states can occur next period, A or B. You observe the following securities:

<table>
<thead>
<tr>
<th>Security</th>
<th>Price</th>
<th>Payoff in state</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

1. Determine the prices of digital securities that pays off in the two states.
2. Determine the current risk free interest rate.
3. What is the current price of a security that pays $20 in state A and $25 in state B?

8.2 Probability [6]
Let $I_s$ be the current price of a digital option that pays 1 if state $s$ occurs. $p_s$ is the time 1 value of investing $I_s$ at time 0, $p_s = I_s(1 + r)$, where $r$ is the one period risk free interest rate. Show that the sum over all states $s$, $\sum_s p_s$, sum to one.

8.3 Digital Options [3]
There are three possible states next period. The risk free interest rate is 5%, and there are two digital options traded, with prices 0.43 and 0.33. What is the price of a digital option for the third state?

8.4 Price [2]
An asset has two possible values next period, $X^u = 50$ and $X^d = 500$. If you are told that the state price probability in the $u$ state is 0.4 and the risk free interest rate is 10%, what is the value of the asset?
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# Chapter 9

## Introduction to Derivatives

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## 9.1 Derivatives

In this chapter we define and show some properties of the “basic” derivatives, namely standard options. While you could be excused for thinking that such obtuse and esoteric financial securities have nothing to do with corporate finance, that would be a mistake. Options are literally “everywhere,” and more and more financial practitioners are using properties of options as tools in their day to day analysis. To justify that statement, consider the following questions:

1. Why do owners of land in downtown L.A. keep their lots undeveloped, reaping only meager dividends from parking fees?

2. What is the nature of payoffs from owning a license to “harvest” manganese on the floor of the Pacific?
3. What is the value of a clause in your mortgage contract that specifies that you can accelerate loan payments any time before maturity and at no penalty?

4. What is the value of the depreciation tax shield if your company at times runs (accounting) losses?

5. Is not equity a call option on the assets of the firm?

6. Is not a convertible bond a straight bond plus a call option on the equity of a firm?

7. Is not default an option to put your assets to the creditors and go on with life?

All of these questions are ones that can be best answered using option pricing theory. An important by-product of our ability to value options is an understanding of what determines the value of an option and the incentives of the option-holders to exploit their rights.

9.2 Definitions

A derivative is a security whose payoff is determined by the value, or payoff, of some other security (usually called the underlying security). The best known examples of derivatives are options. An option is a contingent claim. It has a positive cashflow only if certain events occur. There are several basic types of options, of which some important are:

A Digital Option pays a fixed amount if a specific event occurs.

Example
A digital option that pays $1 if the price of IBM stock 6 months from today is above $500, and nothing otherwise.

A Call Option is an option that gives the owner of the option the right, but not the obligation, to buy a given asset (the underlying) for a given price $K$ (the exercise, or strike price) at some future time $T$ (the option maturity).

Example
A call option that gives you the right to buy one IBM stock at a price of $500 6 months from today.

A Put Option is an option that gives the owner of the option the right, but not the obligation, to sell a given asset for a given price $K$ at some future time $T$. 

Example
A put option that gives you the right to sell one IBM stock at a price of $500 6 months from today.

Options can either be exercised only at the maturity date (European options) or at any time up to and including the maturity date (American options).

9.3 Option Cashflows

The cashflows from put and call options at the maturity date of the option can be summarized as

\[ \text{Call} = \max(0, S_T - K) \]
\[ \text{Put} = \max(0, K - S_T), \]

where \( S_T \) is the price at maturity of the underlying security, and \( K \) is the exercise, or strike price.

To see why this is, let us start by looking at a call option. A call option is the right to buy the underlying at the exercise price \( K \). This possibility is only valuable if the price of the underlying at the exercise date is larger than the exercise price. If the price of the underlying is lower, the option should just be thrown away and the underlying be bought at the then current price. If the price of the underlying at the maturity date is higher than the exercise price, the option is valuable. One can exercise the option, getting the underlying for a price of \( K \), which can then be sold for the market price \( S_T \).

\[
\text{Call} = \begin{cases} 
S_T - K & \text{if } S_T \geq K \\
0 & \text{if } S_T < K 
\end{cases}
\]

Example
Consider the IBM call option with exercise price \( K = 500 \). If the price of an IBM stock at option maturity is 490, the option to buy IBM for 500 has no value, since one IBM stock can be bought for 490 < 500. If the price of IBM is 510, the cash flow from exercising the option, getting one IBM stock, and then selling the stock at a price of 510 is 510 − 500 = 10.

We often summarize the cashflows from the option at maturity in a payoff diagram, a plot of the payoff from holding the option as a function of the price of the underlying. Figure 9.1 shows the payoff diagram for a call option.

The case of a put option is shown similarly

\[
\text{Put} = \begin{cases} 
0 & \text{if } S_T > K \\
S_T - K & \text{if } S_T \leq K 
\end{cases}
\]
Figure 9.1 Payoff diagram for Call Option

Payoff (at maturity)

Figure 9.2 Payoff diagram for Put Option

Payoff (at maturity)
Figure 9.2 shows the payoff diagram for a put option.

As (yet another) piece of terminology, we say that an option is *in the money* if the option would have a positive cashflow if exercised today. The opposite is termed that the option is *out of the money*.

**Example**
A call option is in the money if the current price of the underlying is above the exercise price. A put option is in the money if the price of the underlying is below the exercise price.

### 9.4 Bounds on Option Prices

We are now ready to show some results about option prices. Derivatives are priced relative to the underlying security, and various *bounds* on these prices can be found using *arbitrage*, or *replication* arguments. This is done by finding equivalent ways of achieving the same future payments. If the future payoffs are the same, to avoid arbitrage the value today have to be equal. As notation, use $C_t$ for the call option price at time $t$, and $P_t$ for the put option price at time $t$. Let us first show some obvious bounds.

#### 9.4.1 Positivity

The prices of options have to be greater than or equal to zero.

\[
C_t \geq 0 \\
P_t \geq 0 \\
I_t \geq 0
\]

Why is this obvious? Because the payoff at maturity of an option is always positive or zero, it must cost something to get. Otherwise this is instant money, you get something now (the option price), and then you either get zero or something later, but never have to pay anything. A free lunch...

#### 9.4.2 Simple Upper Bounds

The following are some obvious upper bounds on the values of options.

\[
I_t \leq 1 \\
C_t \leq S_t \\
P_t \leq K
\]
The arguments for these bounds are simple, but they illustrate the typical analysis.

Let us first look at the case of a digital option. Since a digital option is one that pays $1 in only some future states, why should one be willing to pay more than $1 today for the right of possibly receiving $1 in the future?

A call option is simply the right to pay the exercise price \( K \) to receive the underlying security. If the cost of the option is larger than the current value of the underlying, and you would additionally have to pay \( K \) to get the underlying, why not just buy the underlying?

A put option is simply the right to deliver the underlying security and receive \( K \). Why would one be willing to pay more than \( K \) for the right to receive \( K \) in the future?

### 9.4.3 European Call Lower Bound

Let us now show a bound that takes a bit more thinking to realize is the case, namely that the price of a call has to be greater than the current price of the underlying less the present value of the exercise price:

\[
C_t \geq S_t - \frac{K}{(1 + r)^{(T-t)}}
\]

To show this holds is an application of an arbitrage argument. We find two strategies, one of which always have lower or equal future payoffs than the other. The strategy with the potentially higher payoffs will have to have a higher price!

To be specific, consider two strategies, A and B, illustrated in table 9.1. The figure shows the current (time \( t \)) and future (time \( T \)) payoffs for the two strategies.

<table>
<thead>
<tr>
<th></th>
<th>Cash flows</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time ( t )</td>
</tr>
<tr>
<td>A</td>
<td>Buy stock</td>
</tr>
<tr>
<td></td>
<td>Borrow ( PV(K) )</td>
</tr>
<tr>
<td></td>
<td>Total cashflow</td>
</tr>
<tr>
<td>B</td>
<td>Buy call</td>
</tr>
</tbody>
</table>

Table 9.1 Illustrating the European Call lower bound

Strategy A is to buy the underlying and borrow the present value of the exercise price of the option. (We are assuming the possibility of borrowing at the risk free interest rate.) Strategy B is to buy a call option. We can show that strategy B always have equal or higher future cash flow.
To compare the payoff of the strategies at time $T$ it is easiest to split into two cases, depending on whether the option should be exercised.

<table>
<thead>
<tr>
<th>State</th>
<th>$S_T &lt; K$</th>
<th>$S_T \geq K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (borrow to buy underlying)</td>
<td>$S_T - K(&lt; 0)$</td>
<td>$S_T - K(\geq 0)$</td>
</tr>
<tr>
<td>B (buy option)</td>
<td>0</td>
<td>$S_T - K(\geq 0)$</td>
</tr>
</tbody>
</table>

Comparing strategy A with B, strategy B has higher or equal future payoffs, which means strategy B should cost more than strategy A now:

$$C_t \geq S_t - \frac{K}{(1 + r)^{(T-t)}}$$

If this bound did not hold, it would be an arbitrage opportunity, violating the no free lunch assumption.

Putting the two bounds we have found on the price of a call option into one picture, the option price has to be in the hashed area of figure 9.3. We can find similar bounds for puts, but we leave this to a more specialized derivatives text. For the interested some are left as exercises.
9.4.4 Should American Options be Exercised Early?

The difference between American options and European options is that the American ones can be exercised early, before the maturity date, while European options can only be exercised at the maturity date. Since one can always choose to not exercise the American option, an American option can never be less valuable than an European one. But what about the opposite inequality? The opportunity to exercise early sounds valuable, you clearly have more choices. The question is: Will you ever want to use your added opportunities?

The surprising fact is that if the underlying security has no dividends, you should never exercise an American call option early (before the maturity date). Instead of exercising the option, you should keep it. There are two causes of this result. Firstly, if you exercise early, you pay the exercise price. If you had waited till maturity with paying the exercise price, you would have gained the time value of money on holding the exercise price. Secondly, you loose the option value of keeping the option. What if the stock price increases even more?

Thinking about these two causes should also make it obvious that it may be optimal to exercise an American put option early. The argument about time value works the other way with puts. In the case of a put, you get the exercise price, and it is more valuable to get this as early as possible. The time value argues for exercising early, and the option value argument argues for waiting to exercise. There will be cases where the option value is so small that the time value is higher, and it pays to exercise. The simplest case is to look at the case where the underlying has no value.

A Brain Teaser

You are at time $t$ and you contemplate exercising an American call option that is in the money. You know for sure that the price of the underlying at maturity will be less than the exercise price $K$ ($S_T < K$). Your finance professor, as well as your mathematics professor, tells you to not exercise. Wait till the end. They point to the rule above. Your broker says: baloney, exercise now! Who is right?

9.5 Put–Call Parity

As a final illustration of the typical arbitrage arguments, let us show a well known relationship between the prices of put and call options, the price of the underlying and the interest rate.

For European options, and put and call options with the same exercise price
9.5 Put–Call Parity

\[ K \text{ and maturity date } T: \]

\[ C_t = P_t + S_t - \frac{K}{(1+r)(T-t)} \]  

(9.1)

To see this, consider the payoff of the following two strategies:

- Buy a call option.

- Borrow the present value of the exercise price, buy a put option and buy the underlying security.

As table 9.2 illustrates, the payoffs of these two strategies at the maturity date of the options is equal.

<table>
<thead>
<tr>
<th>Table 9.2 Illustrating Put Call Parity</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Buy Call</strong></td>
</tr>
<tr>
<td>(-C_t)</td>
</tr>
<tr>
<td><strong>Buy Put</strong></td>
</tr>
<tr>
<td>(-P_t)</td>
</tr>
<tr>
<td><strong>Buy Stock</strong></td>
</tr>
<tr>
<td>(-S_t)</td>
</tr>
<tr>
<td><strong>Borrow</strong></td>
</tr>
<tr>
<td>(\frac{K}{(1+r)(T-t)})</td>
</tr>
<tr>
<td><strong>Total</strong></td>
</tr>
<tr>
<td>(-P_t - S_t + \frac{K}{(1+r)(T-t)})</td>
</tr>
<tr>
<td><strong>Time (t)</strong></td>
</tr>
<tr>
<td>(S_T &lt; K)</td>
</tr>
<tr>
<td>(S_T - K)</td>
</tr>
<tr>
<td><strong>Time (T)</strong></td>
</tr>
<tr>
<td>(S_T - K)</td>
</tr>
<tr>
<td>(S_T)</td>
</tr>
<tr>
<td>(S_T)</td>
</tr>
<tr>
<td>(-K)</td>
</tr>
<tr>
<td>(-K)</td>
</tr>
</tbody>
</table>

In absence of free lunches the strategies should have the same costs today.

\[ C_t = P_t + S_t - \frac{K}{(1+r)(T-t)} \]

Example

Suppose the underlying security has a price of 50. Put and call options with the same exercise price 50 and time to maturity of 1 month have prices

\[ C_t = 4.50 \]
\[ P_t = 4 \]

If the risk free interest rate \(r\) is 5%, and the underlying security has no cash payments, show how one can make a lot of money!

Let us check put call parity

\[ P_t + S_t - \frac{1}{(1+r)(T-t)} K = 4 + 50 - \frac{1}{(1 + 0.05)^{(1/12)}} 50 = 4.2 \]

This is not equal to the call price of 4.5, put call parity is violated. Here the call option is relatively over-valued, hence you should sell the call and buy the replicating strategy.

The following table illustrates the cash flows from this arbitrage strategy.
By entering into this strategy, we get a payment of 0.3 now, and have no future obligations. This was for one call. To make a lot of money, buy many calls. A clear example of a free lunch (see chapter 2)!

### 9.6 Wrapping Up

As we have seen, the payoff pattern for put and call options, and most other contingent claims, is distinctly *asymmetric*. The CAPM and APT measure risk in terms of variances and covariances. These summarize adequately the risk of symmetric payoffs. But they fail miserably as a measure of the risk of asymmetric payoffs. Therefore, the CAPM and APT are no good to value option-like payoffs, and we need to use the formulation based on *state contingent* cashflows.

### A Brain Teaser

The value of a call increases with the volatility of value of the underlying assets. A consequence of this is that call owners, such as shareholders, have the incentive to raise the volatility of the underlying assets if they can.

### References

#### Original References

Put call parity was first shown in Stoll (1969). Many of the basic option price bounds were first shown in Merton (1973).

#### Textbook references

Cox and Rubinstein (1985) is still a good source, but Hull (2006), Rubinstein (1999) and Cvitanic and Zapatero (2004) are current standard texts on options.
Problems

9.1 XYZ Option [2]
The current price of an American call option with exercise price 50, written on ZXY stock is 4. The current price of one ZXY stock is 56. How would you make a lot of money?

9.2 Put Lower Bound [5]
Show that the following is a lower bound on a put price

\[ P_t \geq \frac{K}{(1 + r)(T-t)} - S_t \]

where \( P_t \) is the current put price, \( K \) is the exercise price, \( r \) is the risk free interest rate, \( (T - t) \) is the time to maturity and \( S_t \) is the current price of the underlying security.

9.3 Options [4]
A put is worth $10 and matures in one year. A call on the same stock is worth $15 and matures in one year also. Both options are European. The put and call have the same exercise price of $40. The stock price is $50. The current price of a (risk free) discount bond (zero coupon bond) paying $1 that matures in one year is $0.90. How do you make risk free profits given these prices?

9.4 Put Upper bound [5]
Show that the following is an upper bound for the price of a put option

\[ P_t \leq \frac{K}{(1 + r)(T-t)} \]

where \( P_t \) is the current put price, \( K \) is the exercise price, \( r \) is the risk free interest rate and \( (T - t) \) is the time to maturity of the option.

9.5 American Put [4]
An American put option with exercise price 50 has a time to maturity of one year. The price of the underlying security has fallen to 10 cents. The risk free interest rate is 5%.
Show that it is optimal to exercise this option early.

9.6 Convexity [8]
Consider three European options written on the same underlying security. The options mature on the same date. The options have different exercise prices \( X_1, X_2 \) and \( X_3 \). Assume \( X_1 < X_2 < X_3 \). All other features of the options are identical.
Let $\omega$ be a number between 0 and 1 satisfying

$$X_2 = \omega X_1 + (1 - \omega) X_3$$

Show that the following inequality must hold to avoid arbitrage (free lunches):

$$C(X_2) \leq \omega C(X_1) + (1 - \omega) C(X_3),$$

where $C(X)$ is the price for an option with exercise price $X$.

9.7 Options [4]
Suppose a share of stock is trading at 30, a put with a strike of 28 is trading at 1 and a call with strike 29 at 8. The maturity of both options is 1 period. The risk free rate is 20%. Is there an arbitrage opportunity (free lunch)?

9.8 MS Option [4]
American call options written on Microsoft's common stock are trading for $8. They carry a strike price of $100, and expire 6 months from today. Microsoft does not pay dividends. At present, its stock price is $104. Hence, the calls are $4 in the money. The annualized six-month risk free rate is 10%. Find an arbitrage opportunity (free lunch) in these numbers and explain how you would exploit it.

9.9 BoA [4]
In-the-money American call options written on BoA's common stock carry a strike price of $55 and expire in 6 months. The annualized six-month risk free rate is 10%. BoA's common stock will go "ex" dividend tomorrow. BoA will pay a $2 dividend. Ms. Johnson holds an option and wonders whether to exercise it. She is worried, because she knows that BoA's stock price will drop tomorrow with $2, making it less likely that her call option will expire in the money. Explain to Ms. Johnson why she should not exercise.
Chapter 10

Pricing Derivatives

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10.1 Pricing Risky Cash Flows

We now want to use the methods for pricing future, uncertain cash flows discussed in chapter 8 to price derivatives. Consider a security that promises a random cash flow $X$ in the future, which equals either a high $X^u$ or a low $X^d$. The cash flows can be illustrated as

\[ P \rightarrow X^u \]
\[ \quad \quad \rightarrow X^d \]

We want to find the value $P$ now of these uncertain, future cash flows. The general rule for pricing was

\[ P = I^u X^u + I^d X^d \]
where \( I^u \) and \( I^d \) are the current market prices for digital options that pay off in the two states. Alternatively we found the value as

\[
P = \frac{1}{1 + r} \left( p^u X^u + p^d X^d \right),
\]

where \( r \) is the risk free interest rate and \( p^u \) and \( p^d \) are time 1 prices of receiving \$1 in states \( u \) and \( d \) (also termed "state price probabilities"). Since \( p^d \) and \( p^u \) sum to one and are never negative, they behave like probabilities. One can alternatively think of the calculation as "The expected payoff under the state-price probabilities," and use the notation

\[
E^*[X] = (p^u X^u + p^d X^d).
\]

to find the following basic pricing formula:

\[
P = \frac{1}{1 + r} E^*[X].
\]

(10.1)

10.2 Pricing Derivatives

This framework is perfect for pricing derivatives, since the values of the underlying securities defines the relevant states.

Example

Stock MNO has price today \( S_0 \) of \$98, and will next period either have a price of \( S^u = 105 \) or a price of \( S^d = 95 \). These two mutually exclusive cases defines all relevant future states for pricing a derivative security written on MNO stock.

\[
\begin{align*}
S^u &= 105 \\
S^d &= 95
\end{align*}
\]

\[
\begin{align*}
S_0 &= 98
\end{align*}
\]

Pricing derivatives merely consist of finding the future cash flows from the derivative as a function of the value of the underlying, and applying the same state price probabilities in the basic pricing relation (10.1). The state price probabilities have to be the same for all derivatives. This follows from the no arbitrage assumption and will be discussed further in chapter 14.
Example
What is the price of an one-period option written on MNO security with exercise price of $100 when the risk free interest rate is 2%?

To evaluate the current price we first have to find the cashflows at the option exercise date. The cash flow of a call option at maturity could be found by looking at the payoff diagram.

Alternatively we could use the formula:

\[ C_T = \max(S_T - K, 0) \]

In this case, the two possible values of the underlying in this case are \( S^u = 105 \) and \( S^d = 95 \). It is only in the case where the price of the underlying is greater than the exercise price of the option that the option has any value.

\[ C^u = \max(0, 105 - 100) = 5 \]

\[ C^d = \max(0, 95 - 100) = 0 \]

To price we need also need the values \( p^u \) and \( p^d \) in the formula:

\[ C = \frac{1}{1 + r} \left( p^u C^u + p^d C^d \right) = \frac{1}{1 + r} \left( p^u \cdot 5 + p^d \cdot 0 \right) \]

Suppose there are traded two digital options, one that pays off $1 if NMO stock is at 105, another that pays off $1 if NMO stock is at 95. The prices of these digital options are \( I^u = 0.47 \) and \( I^d = 0.51 \). We use these prices to find \( p^u \) and \( p^d \):

\[ p^u = I^u (1 + r) = 0.47 \cdot 1.02 = 0.4795 \]
\[ p^d = I^d(1 + r) = 0.51 \cdot 1.02 = 0.5205 \]

and value the option as

\[ C = \frac{1}{1 + r} (p^u C^u + p^d C^d) = \frac{1}{1 + 0.02} (0.4795 \cdot 5 + 0.5202 \cdot 0) \approx 2.35 \]

In general, we calculate the values of one-period call and put options as:

\[ C = \frac{1}{1 + r} E^* \max(S_T - K, 0) \]

\[ P = \frac{1}{1 + r} E^* \max(K - S_T, 0) \]

Example
The price of a put option on MNO with exercise price 100 is found as

\[ P = \frac{1}{1 + r} (p^u P^u + p^d P^d) = \frac{1}{1 + 0.02} (0.4795 \cdot 0 + 0.5202 \cdot 5) \approx 2.601 \]

10.3 Interpreting Equity as an Option

Before you make the mistake of thinking we are only talking about the kind of options traded at the CBOE, let us use option theory to gain some insights in valuing equity. We now treat all share owners as one, and look at the total value of the equity in a company. The equity in a firm can in fact be interpreted as a call option on the assets of the firm!

Consider the simplified balance sheet of a firm, where all numbers are in market values. Today's balance sheet looks like this:

<table>
<thead>
<tr>
<th>Assets (V)</th>
<th>Debt (B)</th>
<th>Equity (E)</th>
</tr>
</thead>
</table>

\( V \) is the value of the assets; \( B \) is the current value of debt; \( E \) is the value of equity. Assuming the debt expires tomorrow, and debt obligations amount to \( D \), the tomorrow's balance sheet is as follows (primes denote tomorrow's values):

| Assets (\( V' \)) | Debt (\( \min(V', D) \)) | Equity (\( \max(0, V' - D) \)) |

Notice that tomorrow's value of equity (which we will denote \( E' \)) is like the payoff on a call option: as long as tomorrow's value of the assets (\( V' \)) is larger than debt obligations (\( D \)), shareholders have a residual claim; if it is not, they receive nothing. Figure 10.1 provides a graphical illustration of the payoff pattern. Hence, today's value of equity (\( E \)) is the value of a call option written on the assets of the firm (whose value equals \( V \)), with strike price equal to \( D \).
Example
Consider the PQZ company. The future value of the firm, $V'$, is either $V_u = 1,000$ or $V_d = 600$. Inclusive of interest payments, the firm's debt amounts to: $D = 750$. This amount is due next period. The price of a digital option that pays 1 in the state when $V'$ equals 1,000 and zero otherwise is 0.45. The riskfree rate $r$ is 10\%.

We will use the by now familiar techniques to value the current value of equity. Use the digital option price to find the "state-price probabilities" $p^u$ and $p^d$.

\[
p^u = 0.45(1 + r) = 0.45 \cdot 1.1 = 0.495.
\]

Since there are only two states, and the probabilities have to sum to one,

\[
p^d = 1 - 0.495 = 0.505
\]

Therefore, the value of equity, $E$, is found as:

\[
E = \frac{1}{1 + r} E'[\max(V' - D, 0)]
\]

\[
= \frac{1}{1 + r} \left( p^u \max(V_u - 750, 0) + p^d \max(V_d - 750, 0) \right)
\]

\[
= \frac{1}{1.1} (0.495 \cdot 250 + 0.505 \cdot 0)
\]

\[
= 112.5.
\]

In fact, we can also apply this approach to valuing the firm. Today's value of the firm is found as

\[
V = \frac{1}{1 + r} E'[V']
\]

\[
= \frac{1}{1 + r} \left( p^u V_u + p^d V_d \right)
\]

\[
= \frac{1}{1.1} (0.495 \cdot 1000 + 0.505 \cdot 600)
\]

\[
= 725.5.
\]
By value additivity, today's value of the corporate debt, $B$, equals:

$$B = V - E = 725.5 - 112.5 = 613.$$ 

Alternatively we can calculate the bond price directly from its cash flows at time 1. The future value of debt, $B'$, is either $B^u = D$ or $B^d = V'$, whichever is lower:

$$B = \frac{1}{1+r} \left( p^u B^u + p^d B^d \right)$$

$$= \frac{1}{1.1} (0.495 \cdot 750 + 0.505 \cdot 600)$$

$$= 613.$$ 

With the calculated bond price, the theoretical yield on the debt is

$$\frac{750}{613} - 1 = 22.35\%,$$

which is far above the riskfree rate of 10%. The debt issue looks like a junk bond. We will return to issues related to corporate debt in chapter 16.

References

Original References

The use of option theory to interpret equity values goes back to at least Black and Scholes (1973)

Textbook References

A standard reference on derivatives is Hull (2006).
Problems

10.1 MLK [4]
The current price of security MLK is 78. Next period the security will either be worth 120 or 90. The risk free interest rate is 33.33%. There are two digital options traded. One pays $1 if MLK is at 120 next period. This option is trading at 0.35. The other pays $1 if MLK is at 90 next period. This option is trading at 0.40.

1. Price a put option on MLK with exercise price $K = 90$.
2. Price a put option on MLK with exercise price $K = 100$.
3. Price a call option on MLK with exercise price $K = 100$.
4. Price a call option on MLK with exercise price $K = 120$. 
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Chapter 11

Pricing of Multiperiod, Risky Investments

Contents

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11.1 Multiple States

We now want to extend the valuation principle in (10.1) to more general cases, with more than one period. Suppose a security promises to pay a random cash flow \( X \) two periods from now, where \( X \) can have one of three possible values: \( X^{uu} \), \( X^{ud} \) or \( X^{dd} \). The payoff pattern can best be represented by a trinomial tree.

![Trinomial Tree Diagram]

\[ P \rightarrow X^{uu} \rightarrow X^{ud} \rightarrow X^{dd} \]
If the one-period interest rate is $r$, then the obvious extension of (10.1) to this more general case, with two periods, would be

$$P = \frac{1}{(1+r)^2} (p^{uu}X^{uu} + p^{ud}X^{ud} + p^{dd}X^{dd})$$

$$= \frac{1}{(1+r)^2} E^*[X],$$

where we have made a slight change in the interpretation of $r$. $r$ is now the riskfree rate per period, and not the rate on a riskfree investment between now and the moment that $X$ is realized. $r$ is assumed to be constant over time. One can easily change that. As usual, the problem reduces to finding state prices $p^{uu}$, $p^{ud}$ and $p^{dd}$, time 2 prices of digital options that pay off in one of the three states.

11.2 Getting to Two States by Adding Time Steps

In the previous section, we introduced three states. The valuation problem can be simplified significantly by introducing an intermediate time step, with the effect that we allow only two states per time step.\(^1\)

To understand this, let $t$ denote time, $t = 0$, $t = 1$ and $t = 2$. $P_t$ is the price of the security at time $t$. At $t = 2$, $X$ is realized and hence $P_2 = X$. We assume that there are only two possible values of $P_1$:

$$P_1 = \begin{cases} P^u_1 & \text{if } X \text{ will be either } X^{uu} \text{ or } X^{ud} \\ P^d_1 & \text{if } X \text{ will be either } X^{ud} \text{ or } X^{dd} \end{cases}$$

It is actually easier to understand this with reference to a graphical picture called a binomial tree, which illustrates the prices and terminal payoffs:

\(^1\)This finessing of the time resolution is at the heart of the famous Black-Scholes formula for option prices.
This picture makes it clear that there is a partial resolution of uncertainty in period 1, one of the three possible states that can occur at time 2 is ruled out, depending on your position in the tree at time 1.

Using the (one period) valuation principle (10.1) to find the prices at time 1, we find

\[ P_1^u = \frac{1}{1 + r} \left( p_{u,u} X_{uu} + (1 - p_{u,u}) X_{ud} \right) = \frac{1}{1 + r} E^* [X|\text{state at } t = 1 \text{ is } u], \]

where \( E^*[X|\text{state at } t = 1 \text{ is } u] \) stands for “the expectation under the state-price probabilities of \( X \) conditional on being in the up-state at \( t = 1 \).” Analogously:

\[ P_1^d = \frac{1}{1 + r} \left( p_{d,u} X_{ud} + (1 - p_{d,u}) X_{dd} \right) = \frac{1}{1 + r} E^*[X|\text{state at } t = 1 \text{ is } d], \]

where \( E^*[X|\text{state at } t = 1 \text{ is } d] \) stands for “the expectation under the state-price probabilities of \( X \) conditional on being in the down-state at \( t = 1 \).”

In the above \( p_{u,u} \) is the state-price probability at time 1 for the state \( X = X_{uu} \). Equivalently, \( p_{u,u}/(1 + r) \) is the time \( t = 1 \) price of a digital option that pays $1 if \( X = X_{uu} \) and zero otherwise. Similarly, \( p_{d,u} \) is the state-price probability at time 1 for the state \( X = X_{du} \). Equivalently, \( p_{d,u}/(1 + r) \) is the time \( t = 1 \) price of a digital option that pays $1 if \( X = X_{du} \) and zero otherwise.

Continuing recursively in time the price at time 0 equals:

\[ P_0 = \frac{1}{1 + r} \left( p^u P_1^u + (1 - p^u) P_1^d \right) = \frac{1}{1 + r} E^*[P_t] \]
Here $p^u$ is the state-price probability at time $t = 0$ for the state $P_1 = P_1^u$. Equivalently, $p^u/(1 + r)$ is the time $t = 0$ price of a digital option that pays (at $t = 1$) $1$ if $P_1 = P_1^u$ and zero otherwise.

Using the time $t = 1$ prices we can find the time $t = 0$ price

$$P_0 = \frac{1}{1 + r} E^* [P_1] = \frac{1}{1 + r} \left( p^u P_1^u + (1 - p^u) P_1^d \right) = \frac{1}{1 + r} \left( p^u \frac{1}{1 + r} \left( p^{u,u} X^{uu} + (1 - p^{u,u}) X^{ud} \right) + (1 - p^u) \frac{1}{1 + r} \left( p^{d,u} X^{ud} + (1 - p^{d,u}) X^{dd} \right) \right) = \frac{1}{(1 + r)^2} E^* \left[ E^* [X | \text{state at } t = 1 \text{ is ...}] \right].$$

For the mathematically sophisticated we note that the last result can be extended. In the absence of free lunches, state-price probabilities can be shown not only to behave as if they were chance numbers, but also as if they were conditional chance numbers. One could then use the "law of iterated expectations" and derive:

$$P_0 = \frac{1}{(1 + r)^2} E^* [E^* [X | \text{state at } t = 1 \text{ is ...}] = \frac{1}{(1 + r)^2} E^* [X],$$

where $E^* [X]$ is the expectation of $X$ at time 0, computed using state-price probabilities obtained from the prices of digital options that pay at time $t = 1$.

It should be clear that the challenge to implementing this strategy is to find the various prices $p^u$, $p^d$, $p^{uu}$, $p^{ud}$ and $p^{dd}$.

### 11.3 A Real-Life Example: Pricing an MCI Call Option

The abstract analysis of the previous example should become clearer when we use a real world example to illustrate the methods. On 11/10/97, MCI's management announced an agreement with WorldCom. Roughly, MCI shares would be exchanged for the equivalent of $51$ in WorldCom stock. At the time, the deal was not a sure thing. The justice department might balk at the monopolizing. Moreover, there was GTE, who had offered $40$ cash per share of MCI, and there was a claim in the Wall Street Journal that GTE was considering an offer of $45$. It would take about five months for the takeover to be finalized. We want to find

---

2The terms of the deal were far more complex - we will return to this example in a later chapter.
the appropriate price for a 4/98 call option on MCI with a $40 strike price. Today's price of MCI stock is $P_{now} = 41.5$. The five-month T-bill rate was roughly 5% (p.a.). Before any of the takeover news hit the market, MCI was selling for $30 a share.

Let us start by ignoring the GTE complication. That leaves two possible states. In April, the price of MCI, $P_{april}$ is either $51$ (the "up" state) if the deal goes through, or $30$, the price the company had before the merger rumors, if everything fails.

\[
\begin{array}{c}
51 \\
41.5 \\
30
\end{array}
\]

Using a technique that we will discuss in the next chapter, we can imply that the market charges a price of $0.5764$ for an option that pays $1$ in the "up" state (and zero otherwise). Therefore, the corresponding state-price probability, $p^u$, equals:

\[
p^u = 0.5764(1.05)^{5/12} = 0.5882.
\]

Using this price we can find the current call price $C_t$ as

\[
C_t = \frac{1}{(1.05)^{5/12}} E^* \left[ \max \left( P_{april} - 40, 0 \right) \right]
\]

\[
= \frac{1}{(1.05)^{5/12}} \left( p^u(51 - 40) + (1 - p^u)0 \right)
\]

\[
= \frac{1}{(1.05)^{5/12}} 0.5882 \cdot 11
\]

\[
= 6.34.
\]

As a matter of fact, at the close of 11/10 an MCI 4/98-$40 call was selling for 4 5/8. This is below the price we just estimated. Maybe we should be buying these cheap options? Before we conclude that this is a profit opportunity, we must at least consider the possibility that we have made a mistake in our pricing above. Remember that we ignored the added complication of GTE's offer. Let us show how we can modify the analysis to account for this.

Let us plausibly posit that by January, we will know whether the deal goes through. If the deal fails, there is still GTE, who is prepared to offer $40-$45
a share. As a reasonable estimate of the final bid from GTE, let us take the midpoint of the GTE offers, $42.5. So, by April, we have the following three possible outcomes: $51, $42.5 or $30. The first occurs for sure if it is known by January that the WorldCom takeover gets an "OK". The second or third outcomes are assumed to occur randomly if the deal is called off by January.

Graphically, this is what we have in mind:

```
51

41.5

42.5

30
```

The usual method for attacking a problem like this is to work backward. Based on the outcomes at the terminal nodes (here April), we first find prices at time 1 (here January), and then use the time 1 prices to find the time 0 prices (now).

Let us start with the case where the deal goes through, it is announced in January that the WorldCom deal will be finalized. In that case, the payoff in April is a sure thing, and we find the January price $C_{January}^{OK}$ as

$$C_{January}^{OK} = \frac{1}{(1 + \frac{r}{12})^{3/12}} (51 - 40) = \frac{1}{1.053^{1/12}} 11 = 10.87.$$ 

If the deal does not go through, we are left with two possible outcomes at time 2, in January, and we want to find the price $C_{January}^{notOK}$.

$$C_{January}^{notOK} = \frac{1}{(1 + \frac{r}{12})^{3/12}} \left( p_{January}^{d,u} (42.5 - 40) + \left( 1 - p_{January}^{d,u} \right) 0 \right),$$

where $p_{January}^{d,u}$ denotes the January state-price probability for the event that GTE takes over MCI for $42.5 per share. For now we will not discuss how we got the number, but market prices in November did imply a $0.7790 contingent January price for a digital option that pays $1 when GTE manages to purchase MCI by April for $42 1/2. The digital option price is contingent upon the event that the WorldCom deal would break down. We will discuss in Chapter 12 how to get this price. Hence,

$$p_{January}^{d,u} = 0.7790(1 + \frac{r}{12})^{3/12} = 0.7886.$$ 

Using these prices, we find

$$C_{January}^{notOK} = \frac{1}{1.053^{1/12}} (0.7886 \cdot 2.5 + (1 - 0.7886)0) = 1.95.$$
Given the two possible prices at time 1 (January), we can work further backwards in time to find the time 0 (November) value of the April call:

\[
C = \frac{1}{(1+r)^{2/12}} (p^u C_{\text{January}}^{\text{OK}} + (1-p^u) C_{\text{January}}^{\text{notOK}})
\]

To find prices we need \( p^u \), the state-price probability for the January event that the WorldCom takeover goes through. For now we will just state that this can be calculated as

\[
p^u = 0.2239
\]

We will get back to how this number can be estimated from market prices. Given the price, we calculate the November value of the call option as

\[
C = \frac{1}{(1+r)^{2/12}} (0.2239 \cdot 10.87 + (1 - 0.2239) \cdot 1.95) = 3.92.
\]

That is below the market price of $4 5/8 ($4.63), but the $0.71 difference is not necessarily sufficient to decide to write MCI April call options (one has to incorporate transaction costs and the cost of margin when writing options. In addition, the $4 5/8 price is a closing price, which could well be an ask price. The bid price - relevant when one shorts - may easily be 5% below the ask).

To illustrate the general principle that the state-price probabilities can be used to price any derivative, including the underlying security itself, we can verify the November price of MCI of $41 1/2.

We will do so for the calculation where we account for the GTE offer (involving three possible states in April). We work backwards, and start with the most favorable January event: it is known that the WorldCom deal will be finalized. In that case, the January price of MCI, \( P_{\text{January}}^{\text{OK}} \), is

\[
P_{\text{January}}^{\text{OK}} = \frac{1}{(1+r)^{3/12}} 51 = \frac{1}{(1.05)^{3/12}} 51 = 50.4.
\]

If the deal does not go through, one find the January price as

\[
P_{\text{January}}^{\text{notOK}} = \frac{1}{(1+r)^{3/12}} \left( p_{\text{January}}^{d,u} 42.5 + (1 - p_{\text{January}}^{d,u}) 30 \right)
\]

\[
= \frac{1}{1.05^{3/12}} (0.7886 \cdot 42.5 + (1 - 0.7886)30)
\]

\[
= 39.4.
\]
Going back in time, the November price of MCI is calculated as

\[ P_{\text{now}} = \frac{1}{(1 + r)^{2/12}} \left( p^u P_{\text{January}}^{\text{OK}} + (1 - p^u) P_{\text{January}}^{\text{notOK}} \right) \]

\[ = \frac{1}{(1 + r)^{2/12}} \left( p^u \times 50.4 + (1 - p^u) \times 39.4 \right) \]

\[ = \frac{1}{1.05^{2/12}} \left( 0.2239 \times 50.4 + (1 - 0.2239) \times 39.4 \right) \]

\[ = 41.5 \]

This confirms the market price.

11.4 General Strategy

Our valuation principle (10.1) really calls for the following exercise.

1. Determine all possible future "states" (scenarios) - a state is an event where the security pays a particular quantity.

2. If there are more than two possible states, add intermediate branches to make a tree with only two possible branches at each node.

3. Determine the price that the market charges for each state at each node in the tree and convert those state prices to a state-price probabilities by multiplying them with the riskfree rate. (In fact, one can directly derive the state-price probabilities.)

4. Working backward in time, compute the price of the security at each node as the expected payoff (using the state-price probabilities), discounted at the riskfree rate.

The main problems in implementing the pricing strategy above is to find the relevant states, and then, given the states, to find prices. This then is the topic of the next chapter, how to get states and how to use market prices to imply state-price probabilities.

References

Original References

The basic reference for the existence of this kind of multiple period, contingent expectation formulation is Harrison and Kreps (1979).
Problems

11.1 **MLK** [5]
The current price of security MLK is 89.50. Two periods from now the security will either be worth 120, 100 or 90. The one period risk free interest rate is 8.45%. There are three digital options traded. One pays $1 if MLK is at 120 two periods from now. This option is trading at 0.35. The second pays $1 if MLK is at 100 two periods from now. This option is trading at 0.25. The third pays $1 if MLK is at 90 two periods from now. This option is trading at 0.25.

1. Price a (two period) put option on MLK with exercise price $K = 90$.
2. Price a (two period) put option on MLK with exercise price $K = 100$.
3. Price a (two period) call option on MLK with exercise price $K = 100$.
4. Price a (two period) call option on MLK with exercise price $K = 120$.  
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Chapter 12

Where to Get State Price Probabilities?

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12.1 Implementation

As discussed in the previous chapter, to implement pricing we first have to determine the relevant future states and then to find the state prices. In this chapter we first show a general strategy for generating future states, and then discuss how we can use market prices for traded securities to infer state prices, given the set of possible states.

12.2 Generating the Possible Future States

In the case of derivatives, states are determined by the possible future values of the underlying security. In many cases it is clear what these can be. The MCI example in the previous chapter was one of these.
In most cases it is much harder to find these possible future values, and it is necessary to find a way of generating the possible states. What we want is to find a range of values that we think are likely outcomes. This means we need to think about probabilities for changes in the value or the underlying, which is measured by the variability, or volatility, of the underlying.

Example
You have two securities with the following possible payoffs:

\[
\begin{align*}
100 & \quad 110 \\
90 & \quad 100
\end{align*}
\]

\[
\begin{align*}
150 & \quad 50
\end{align*}
\]

The two outcomes are equally likely. The first security is clearly less variable than the second, which is confirmed by calculating the standard deviation, or volatility, of the return of two securities:

\[
\begin{align*}
E[r_A] &= \frac{1}{2} \frac{110 - 100}{100} + \frac{1}{2} \frac{190 - 100}{100} = 0 \\
\sigma^2(r_A) &= \frac{1}{2} (-0.1)^2 + \frac{1}{2} (0.1)^2 = 0.01 \\
\sigma(r_A) &= \sqrt{0.01} = 0.1 = 10\% \\
E[r_B] &= \frac{1}{2} \frac{150 - 100}{100} + \frac{1}{2} \frac{50 - 100}{100} = 0 \\
\sigma^2(r_B) &= \frac{1}{2} (-0.5)^2 + \frac{1}{2} (0.5)^2 = 0.25 \\
\sigma(r_B) &= \sqrt{0.25} = 0.5 = 50\%
\end{align*}
\]

A useful way of generating possible states relies on the volatility of underlying security. Let \( \sigma \) denote the volatility (standard deviation) of the return on the underlying asset (expressed in percent per period). Let \( \mu \) denote the average of the return. If the time-0 value of the asset is \( V_0 \), then at \( t = 1 \), we choose the two possible values for \( V_1 \) as

\[
V_1 = \begin{cases} 
V_0 e^{\mu + \sigma} \\
V_0 e^{\mu - \sigma}
\end{cases}
\]

In a picture
If one wants three possible states at $t = 1$, then one introduces an intermediate step at $t = 1/2$, and sets $V_{1/2}$ to be either

$$V_{1/2} = \begin{cases} 
V_0 \exp\left(\frac{1}{2} \mu + \sigma \sqrt{\frac{1}{2}}\right) \\
V_0 \exp\left(\frac{1}{2} \mu - \sigma \sqrt{\frac{1}{2}}\right) 
\end{cases}$$

$V_1$ then becomes

$$V_1 = \begin{cases} 
V_{1/2} \exp\left(\frac{1}{2} \mu + \sigma \sqrt{\frac{1}{2}}\right) \\
V_{1/2} \exp\left(\frac{1}{2} \mu - \sigma \sqrt{\frac{1}{2}}\right) 
\end{cases}$$

That is three possible cases:

$$\begin{cases} 
V_0 \exp\left(\mu + \sigma 2 \sqrt{\frac{1}{2}}\right) \\
V_0 \exp\left(\mu\right) \\
V_0 \exp\left(\mu - \sigma 2 \sqrt{\frac{1}{2}}\right) 
\end{cases}$$

This generates a tree with two possible branchings at each node:
In general, to generate $N + 1$ possible values at $t = 1$, $N - 1$ intermediate steps have to be introduced, and the value between the $n$'th and $(n + 1)$'th step changes as follows:

$$V_{n+1}^w = \begin{cases} V_{n}^u \exp\left(\frac{\mu}{N} + \sigma\sqrt{\frac{1}{N}}\right) \\ V_{n}^d \exp\left(\frac{\mu}{N} - \sigma\sqrt{\frac{1}{N}}\right) \end{cases}$$

You should check that this produces $N + 1$ values at time $t = 1$.

**Example**

If IBM stock has an average return of 15% per year, and a volatility of 30% per year, and its value is $100$ now, then we could value one-month options on IBM as follows. In a rough, two-state analysis, we would take the possible IBM prices at maturity to be one of the following two:

$$\begin{align*} 
100 \exp\left(\frac{0.15}{12} + 0.30\sqrt{\frac{1}{12}}\right) &\approx 110.4 \\
100 \exp\left(\frac{0.15}{12} - 0.30\sqrt{\frac{1}{12}}\right) &\approx 92.8
\end{align*}$$

In a more sophisticated analysis, with 5 possible end-values, we introduce three intermediate steps at 1/4 month intervals (roughly one week). The possible values at maturity would be:

$$\begin{align*} 
100 \exp\left(\frac{0.15}{12} + 0.30 \cdot 4\sqrt{\frac{1}{48}}\right) &\approx 120.40 \\
100 \exp\left(\frac{0.15}{12} + 0.30 \cdot 2\sqrt{\frac{1}{48}}\right) &\approx 110.40 \\
100 \exp\left(\frac{0.15}{12}\right) &\approx 101.26 \\
100 \exp\left(\frac{0.15}{12} - 0.30 \cdot 2\sqrt{\frac{1}{48}}\right) &\approx 92.86 \\
100 \exp\left(\frac{0.15}{12} - 0.30 \cdot 4\sqrt{\frac{1}{48}}\right) &\approx 85.15
\end{align*}$$

There are more sophisticated ways to obtain a description of the states, but these must be dealt with in more advanced finance texts.

### 12.3 Now the State Price Probabilities

The next problem is to find probabilities given the set of possible states. These are actually estimated from market prices. Remember that all derivatives written on the same security can be priced using the same state price probabilities

$$P = \frac{1}{1 + r} \left( p_u X^u + p_d X^d \right) = \frac{1}{1 + r} \left( p_u X^u + (1 - p_u) X^d \right)$$

When there are only two states, the price of the underlying security is enough to determine the state price probability $p_u$. Alternatively we can use the price of any other derivative of the same underlying security, such as an call option, a put option or a futures contract.
Example
Let us return to the PQZ company of section 10.3. The future value of the firm, $V$, was either $V^u = 1,000$ or $V^d = 600$, and the firm had debt of $D = 750$ due next period. We calculated the current value of equity as

$$112.5 = E = \frac{1}{1+r} \left( p^u \max(V^u - 750, 0) + (1 - p^u) \max(V^d - 750, 0) \right)$$

$$= \frac{1}{1.1} \left( p^u \cdot 250 + (1 - p^u) \cdot 0 \right)$$

If we know the current value of equity is $112.5$, it is a simple matter to solve the above equality for the implied state price probability $p^u$:

$p^u = 0.495$,

which is what we used to find the value of equity.

The market price of one security, equity in the example, is thus enough to find the probabilities $p^u$ and $p^d$. This easy estimation of implied probabilities is in fact a major reason for the use of the formulation with state price probabilities instead of the one involving digital options.

In general, if there are $N + 1$ possible outcomes, one constructs a tree with $N - 1$ intermediate steps and $n + 1$ branchings at step $n$ ($n = 0, 1, 2, ..., N - 1$). For each of these branchings, there will be a state price probability of an up-state and one for a down-state. In such a multi-period problem we need the prices of several derivatives in order to solve for all those state price probabilities. Unless, of course, one assumes that the state price probabilities remain constant across all branchings. The latter is what most people do in practice.

12.4 Pricing Call Options: The MCI Example Again

Let us return to the MCI example of section 11.3. How did we get the state prices in that example?

Example
First consider the two-state case, where the following outcomes were possible.
If the November price of MCI stock is $41.5, the state price probabilities implicit in the market's pricing of MCI stock is found as:

$$P_{\text{now}} = \frac{1}{(1 + r)^{12/12}} E^{u} [P_{\text{April}}]$$

$$41.5 = \frac{1}{1.05^{3/12}} (p^u 51 + (1 - p^u) 30).$$

Solving for $p^u$:

$$p^u = 0.5882.$$

As we see, using observed prices in the market, it is possible to get the market's implied probabilities.

This approach generalizes to cases where there is more than two possible states, but it is necessary to have more information.

Example

Let us go back to the MCI example of section 11.3 where there was also an cash offer by GTE. Suppose that in addition to knowing the November price of MCI stock, $41 1/2, we also know the price of a January MCI call option with strike $42 1/2. This option closed on 11/10 at $1 3/4. We again use the recursion trick, and start with the most favorable January event: it is known that the WorldCom deal will be finalized. In that case the outcome in April is a sure thing ($51). Hence the January price of MCI, $P_{\text{January}}^{OK}$, is

$$P_{\text{January}}^{OK} = \frac{1}{(1 + r)^{3/12}} 51 = \frac{1}{(1.05)^{3/12}} 51 = 50.4.$$

In the other January state when the WorldCom deal does not go through, the price, $P_{\text{January}}^{\text{notOK}}$, would be

$$P_{\text{January}}^{\text{notOK}} = \frac{1}{(1 + r)^{3/12}} \left( p^d_{\text{January}} 42.5 + (1 - p^d_{\text{January}}) 30 \right).$$

Going back in time, the November price of MCI should be:

$$P_{\text{now}} = \frac{1}{(1 + r)^{2/12}} \left( p^u P_{\text{January}}^{OK} + (1 - p^u) P_{\text{January}}^{\text{notOK}} \right).$$
12.5 State Price Probabilities and True Probabilities

Now consider the January option with strike 42 1/2. Using the general pricing formula (10.1), its value, $C$, is:

$$
C = \frac{1}{(1 + r)^{2/12}} \left( p_u \max (P_{January}^{OK} - 42.5, 0) + (1 - p_u) \max (P_{January}^{notOK} - 42.5, 0) \right)
$$

because, for sure:

$$
P_{January}^{notOK} < 42.5.
$$

We can imply $p_u$ from $C = 1.75$. It equals 0.2239. Given $p_u = 0.2239$, we can imply $p_{January}^{u}$ from $P_{now} = 41.50$. We get: $p_{January}^{u} = 0.7886$.

12.5 The Relationship between State Price Probabilities and True Probabilities

Until now, we have not discussed the relationship between state price probabilities and true probabilities. The truth of the matter is that in general, there is no relationship. Except for the obvious case when a state has zero chance of occurring. In that state the state price and the state price probability would be zero as well. (Who would be willing to pay for a digital option that pays in a state with zero probability of occurring?)

If all market participants were risk neutral, state price probabilities and true probabilities would be the same (provided markets are efficient). That is because assets would then be priced as the (true) expectation of the future payoff, discounted at the riskfree rate. Formula (10.1) must hold at the same time, so true probabilities and state price probabilities coincide.

When markets reflects risk aversion, state price probabilities are like colored chance numbers, probabilities biased by fear. "Bad states" (states when investors are all poor) will carry a state price probability which is above the true probability that the state really occurs. Conversely, states where investors will be rich carry a lower state price probability than the true probability would reflect.

Example

Let us apply this to the MCI case. In the three-state analysis, we computed the state price probability of the "up" state (when WorldCom announces by January that the takeover is OKed on all fronts) to be about 22%. Does that mean that the market thinks there is only a 22% probability that the WorldCom takeover will go through? Presumably not. The true probability is colored by the fact that investors will be very rich if that state occurs. Why? The WorldCom takeover is most likely to succeed if the WorldCom stock price is high,
which would only occur when there is a general increase in stock prices. Because WorldCom takes over MCI only in a state when everybody is rich anyway, the corresponding state price probability is lower than the true probability. The chance of WorldCom taking over MCI may be as high as 1/3.

On 11/10/97, a one in three chance looked awfully low when compared with the stories in the newspapers. But that is what the market was signaling!

Just to get the semantics right. State price probabilities originally were referred to as risk-neutral probabilities. We want to avoid that terminology, because it is confusing, as if risk neutrality held. The correct reasoning is rather the "valuation would be consistent with risk neutrality if the true probabilities were equal to the state price probabilities." The ratio of state price probabilities to true probabilities is referred to in "rocket science" as the state price density. "Rocket scientists" also use another word for the collection of state price probabilities (probably to keep normal human beings out of the business), namely the equivalent martingale measure.

A final, important remark concerns state price probabilities in the famous Black-Scholes model, of which you may have heard. We will explain later in this book how the Black-Scholes model relates to our two-state model. At this point, it suffices to mention that the Black-Scholes model at first looks like a counter-example to the proposition that there is really no relation between the physical probabilities and the state price probabilities. Specifically, the true volatility of the price of the underlying asset (i.e., the volatility under the physical probabilities) is the same as the volatility under the state price probabilities. That is, the volatility carries over from the physical probabilities to the state price probabilities. So, it looks like more things carry over besides zero-probability events. That's not exactly correct. The Black-Scholes world is very specific. In particular, price paths are continuous, yet not predictable (up to drift). This enlarges the possible events of zero probability and is the reason why the true volatility can be used to price securities. In other words, the commonality of the volatility under the physical and state price probabilities is an artifact of the Black-Scholes world, and certainly does not generalize. It does not apply, for instance, in our two-state world.

12.6 The Power of State Price Probabilities – Even If You Don’t Know Them

It is true that it may be difficult at times to infer the state price probabilities. But the fact that they exist (absent free lunches), and hence, that prices are (discounted) expectations means that you can immediately determine when prices
are unreasonable. Indeed, you will be able to recognize arbitrage opportunities by
the mere finding that prices violate basic properties of mathematical expectation.
This is extremely powerful, as the following example shows.

Example
A popular model in textbooks on fixed-income securities pricing assumes that the term
structure of interest rates is flat. That is, \( r_t \), the \( t \) period interest rate, does not change
with \( t \). Yet at the same time interest rates are allowed to change randomly. This combination
of a flat term structure but random interest rates generates prices that violate one of the
basic principles of mathematical expectation, namely, Jensen’s inequality. This inequality
implies, among other things, that for a nontrivial random variable \( X \) and any exponent \( \gamma > 1 \),

\[
E^*[X^\gamma] > (E^*[X])^\gamma.
\]

(We take expectations with respect to the state price probabilities, but the property holds
for any probabilities.)

Now consider the following situation. An announcement is about to be made about
interest rate changes. Let \( r_t \) denote the new level of interest rates, after the announcement.
The term structure will be flat: \( r_t = r \), all \( t \). Just before the announcement, the term
structure is also flat; interest rates \( r^\sim_t \) are equal to some constant \( r^\sim \).

The prices of one-period and two-period pure discount bonds are related in a simple way,
both before and after the announcement. Before the announcement:

\[
P_2^\sim = \frac{1}{(1 + r^\sim)^2} = \left( \frac{1}{1 + r^\sim} \right)^2 = (P_1^\sim)^2,
\]

where \( P_2^\sim \) and \( P_1^\sim \) denote the pre-announcement two-period and one-period pure discount
bond prices, respectively. After the announcement,

\[
P_2 = \frac{1}{(1 + r)^2} = \left( \frac{1}{1 + r} \right)^2 = (P_1)^2,
\]

where \( P_2 \) and \( P_1 \) denote the post-announcement two-period and one-period pure discount
bond prices, respectively.

At the same time, prices ought to be discounted expectations of future values. Expecta­
tions are to be computed with respect to state price probabilities. In particular,

\[
P_2^\sim = E^*[P_2]; \quad P_1^\sim = E^*[P_1].
\]

(We can ignore discounting because the announcement of interest rate changes is about to
be made; the time value of money over the short announcement period is zero.)

Now use the relationships between \( P_2^\sim \) and \( P_2 \), and between \( P_1^\sim \) and \( P_1 \), to conclude that:

\[
(P_1^\sim)^2 = P_2^\sim = E^*[P_2] = E^*[P_1]^2 > (E^*[P_1])^2 = (P_1)^2.
\]

The inequality is imposed by Jensen’s inequality. Note the contradiction: \( (P_1^\sim)^2 > (P_1)^2 \)?
Because of this contradiction, the proposed pricing structure, namely, a flat but random term structure, is inconsistent with the proposition that prices are discounted expectations of future values. But this means that the proposed pricing structure implies arbitrage. If there were no arbitrage opportunities, then prices would be discounted expectations, and they would not generate inconsistencies when evaluated using Jensen’s inequality.

Notice that we derived this conclusion without having to know what the state price probabilities were. We merely had to know that they existed. This illustrates the power of our approach. And it explains why Wall Street nowadays is awash with mathematicians who know all about mathematical expectation.
Problems

12.1 *MOP* [4]
The risk free interest rate is 4%. The current price of a stock in company MOP is 50. Next period the price of MOP stock will either be 40 or 90.

- Determine the state price probabilities for the two states.
- Determine the current price for a digital option that pays one dollar when the MOP price is 40 next period.
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Chapter 13

Warrants

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13.1 Definition

We will now apply our general pricing principles to another corporate security, the warrant. Like an equity call option, a warrant entitles its holder to purchase shares at a pre-specified strike, or exercise, price \( K \). The difference between a standard option and the warrant is that when a warrant is exercised, new equity is issued by the firm, and this equity is purchased from the company itself at the warrant exercise. Payment of the strike price leads to a cash inflow for the company. This cash inflow leads to an increase in the value of the firm.

13.2 Firm Value and Warrants

The situation is as follows. Let \( W \) denote the total value of warrants outstanding and \( n_w \) the number of warrants outstanding. Let there be \( n \) shares before warrant exercise. We will assume that one warrant entitles the holder to purchase one share.

Before warrant exercise, we have equity value \( (E) \), warrant value \( (W) \) and bond value \( (B) \). Because of value additivity and no free lunches, this should all
add up to the value of the firm, \( V \):

\[ V = E + W + B. \]

Immediately after warrant exercise, we have new equity value \( (E') \), and new bond value \( (B') \). Because of value additivity, this should all add up to the new value of the firm, \( V' \), which equals:

\[ V' = V + Kn_w. \]

Hence:

\[ V' = E' + B'. \]

The original and after-exercise per-share value, \( s \) and \( s' \), respectively, equal

\[ s = \frac{E}{n} \]

and

\[ s' = \frac{E'}{n + n_w}. \]

It looks like there is dilution: The original shareholders have to share the value of the firm with new shareholders. The latter would exercise their warrants only if \( s' \geq K \), so they add less to the firm’s value than they get. The situation is a bit more complex. Just before exercise,

\[ s = s', \]

because otherwise there would be a free lunch. There is actually no dilution. In fact, if there were any dilution, it should already have been incorporated in the equity value \( E \) upon announcement of the issue of warrants! That is what efficient markets are all about...

There is also an issue as to when one should exercise warrants. This is because the exercise changes the value of the firm. A full strategic analysis is really the subject of advanced corporate finance, and requires an appeal to (cooperative and/or noncooperative) game theory. We will discuss only one specific case below. But it is clear that changes in the exercise strategy alters the value of the warrants, and hence, of all corporate liabilities. Analysis of exercise strategies is therefore a prerequisite to sound valuation of warrants. We will encounter a similar case of how strategic considerations can affect valuation when we discuss convertible bonds in Chapter 16.
13.3 Valuation

Let us show an example of valuation of warrants. For simplicity assume the company has no debt. All warrantholders are assumed to exercise at the same time. The condition for exercise is \( s' - K > 0 \), i.e.

\[
\frac{V + Kn_W}{n + n_W} - K > 0.
\]

The gain from exercising a warrant if all warrants are exercised simultaneously is:

\[
\frac{V + Kn_W}{n + n_W} - K.
\]

Using the fundamental pricing formula (10.1), the per-warrant value is simply the expected future payoff (based on the state-price probabilities), discounted at the riskfree rate:

\[
\frac{W}{n_W} = \frac{1}{1 + r} E^* \left[ \max \left( \frac{V + Kn_W}{n + n_W} - K, 0 \right) \right].
\]

We can rewrite this as follows:

\[
\frac{W}{n_W} = \frac{n}{n + n_W} \frac{1}{1 + r} E^* \left[ \max \left( \frac{V}{n} - K, 0 \right) \right].
\]

We will return to warrant–like issues in Chapter 16, and ask why companies issue warrants and convertible securities if these are so difficult to value.

References

Textbook References

The pricing of warrants using options pricing theory is discussed in Cox and Rubinstein (1985).
Problems

13.1 *Option/Warrant* [2]
Consider a warrant and a call option, both written on IBM stock,

(a) Which of these securities has been issued by IBM?

Consider two scenarios.
(1) You own a warrant on IBM with maturity 6 months and exercise price 100.
(2) You own a call option on IBM with maturity 6 months and exercise price 100.

(b) Will you be indifferent between these two alternatives?

(c) If not, which one would you prefer?

13.2 *Warrants* [6]
A firm has issued 500 shares of stock, 100 warrants and a straight bond. The warrants are about to expire and all of them will be exercised. Each warrant entitles the holder to 5 shares at $25 per share. The market value of the firm’s assets is 25,000. The market value of the straight bond is 8,000. That of equity is 15,000.

1. Determine the post-exercise value of a share of equity.

2. What is the mispricing of equity?

3. From the proceeds of immediate exercise, value the warrant.

4. Now assume that the market value of debt becomes $9,000, to reflect the increase in the value of the firm upon warrant exercise (which lowers the probability of bankruptcy). Re-compute the value of the warrants and of equity.
Chapter 14

The Dynamic Hedge Argument

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14.1 Pricing

In this chapter we will return to and go more deeply into some of the remaining issues we have so far skipped lightly over. We will show that the state prices must be the same for all derivatives. We will also illustrate in more detail exactly how state prices are set, in the setting of the binomial option pricing model.

14.2 Why State Prices Must be Equal

We will first argue that state prices must be the same for all derivatives written on the same underlying security. To do this we make a simple arbitrage argument: if state prices are not the same, it will be possible to create a free lunch.

Example

To illustrate we will look at a two-state case. The future value $V$ of the underlying asset can be either of 100 or 50:
The value of the underlying is assumed to satisfy the fundamental valuation formula (10.1) with an "up-state" state-price probability $p^u$ of 0.5, and a risk free interest rate of 10%, implying

$$V_0 = \frac{1}{1+r} \left( p^u V^u + (1-p^u) V^d \right) = 68.2.$$

Suppose there is a call trading with strike 75. This call is priced at 13.6. By plugging this into the pricing relation

$$C_t = \frac{1}{1+0.1} \left( p^u \max(100 - 75, 0) + (1-p^u) \max(50 - 75, 0) \right)$$

$$= \frac{1}{1.1} \left( p^u 25 + (1-p^u) 0 \right)$$

$$= 13.6.$$

and solving for $p^u$, we find that this option satisfies the risk neutral pricing relation (10.1), but with a different "up-state" state price probability, namely $p^u = 0.6$.

We can construct a riskfree portfolio by purchasing one unit of the underlying asset and selling short 2 units of the call. In the "up state", this portfolio would earn

$$100 - 2 \cdot 25 = 50.$$

In the "down state" it would earn the same! Since any riskfree asset should earn 10%, the value of this portfolio should be:

$$\frac{1}{1+r} 50 = 45.5.$$

Yet its actual value is:

$$68.2 - 2(13.6) = 41.0.$$

This portfolio makes a sure $50 with a $41 investment, a 22% return. This is a bit above the risk free rate of 10%...

A clear free lunch...

As the example illustrates, if the state price probabilities are not consistent across derivatives, it is possible to create arbitrage opportunities, or free lunches. Since it is an axiom of finance that free lunches do not exist, state price probabilities must be consistent.
14.3 The Binomial Option Pricing Model

Constructions like the one above was first used to derive the famous "binomial option pricing model." Let us discuss that model now, and use it to give an explicit derivation of state price probabilities.

The basic assumption of the binomial option pricing model is on the evolution of the price of the underlying security. We introduce constants $u$ and $d$ which are used to generate the possible future values of the underlying as follows.

\[ S^u = uS \]
\[ S^d = dS \]

**Example**

The current price of the underlying is 100. The underlying will next period be one of 110 or 90.

\[ uS = 110 \]
\[ S_0 = 100 \]
\[ dS = 90 \]

What are the values of $u$ and $d$?

\[ S^u = 110 = uS_0 = u100, \quad u = \frac{110}{100} = 1.1 \]
\[ S^d = 90 = dS_0 = d100, \quad d = \frac{90}{100} = 0.9 \]

The constants $u$ and $d$ can be thought of as "jump intensities." The difference between $u$ and $d$ is related to the volatility of the underlying security. To avoid arbitrage, need $d < 1 + r < u$, where $r$ is the riskfree rate.

For concreteness, let us value a call option with exercise price $K$. Given the evolution of the underlying security, the value of the call will depend on the outcome of the underlying.
The Dynamic Hedge Argument

\[ C_u = \max(0, S^u - K) \]
\[ C_d = \max(0, S^d - K) \]

Example

In the above example, with

\[ uS = 110 \]
\[ S_0 = 100 \]
\[ dS = 90 \]

what are the payoffs at time 1 of a one-period call option with exercise price \( K = 100 \)?

\[ C_u = \max(0, 110 - 100) = 10 \]
\[ C_d = \max(0, 90 - 100) = 0 \]

To price a call option consider the following question: Can we construct a riskfree portfolio by combining one option with \( m \) units of the underlying stock?

Example

To get a risk free portfolio, the payoffs have to be equal in the two states:

\[ C_u + mS^u = C_d + mS^d \]

Solving for \( m \):

\[ C_u - C_d = m(S^d - S^u) \]
Buying one call option and selling a half of the underlying short will produce a portfolio that is risk free, with cost equal to \( C_0 - \frac{1}{2} S_0 \):

\[
\begin{align*}
C^u - \frac{1}{2} S^u &= 10 - \frac{1}{2} 110 = -45 \\
-C_0 + \frac{1}{2} 100 \\
C^d - \frac{1}{2} S^d &= 0 - \frac{1}{2} 90 = -45
\end{align*}
\]

In general, it is always possible to construct a portfolio with a risk free payoff next period by choosing

\[
m = \frac{C^u - C^d}{S^d - S^u} \approx \frac{10 - 0}{90 - 110} = \frac{-1}{2}
\]

Since it is riskfree, the value of the portfolio should be the present value of its payoff:

\[
(C_0 + mS_0) = \frac{1}{1 + r} (C^u + mS^u)
\]  

(14.2)

One can alternatively argue: The amount invested should only earn the risk free interest rate.

\[
(C_0 + mS_0)(1 + r) = (C^u + mS^u)
\]  

(14.3)

By plugging in the value of \( m \) given in (14.1) in either of (14.2) or (14.3) and simplifying we get the following expression for the call price

\[
C_0 = \frac{1}{1 + r} \left( \left( \frac{1 + r - d}{u - d} \right) C^u + \left( 1 - \left( \frac{1 + r - d}{u - d} \right) \right) C^d \right)
\]

If we define

\[
q = \left( \frac{(1 + r) - d}{u - d} \right)
\]

and plug in, the expression should become familiar

\[
C_0 = \frac{1}{1 + r} \left( q C^u + (1 - q) C^d \right)
\]
Notice that, since we have assumed \( d < (1 + r) < u \),

\[
0 < q = \frac{(1 + r) - d}{u - d} < 1,
\]

so it is a probability. In fact, it is the state-price probability we have been talking about. In other words:

\[
p^{u} = q = \frac{(1 + r) - d}{u - d}.
\]

**Example**

In the previous example, if the one period interest rate is 5%, calculate the price of the call:

\[
p^{u} = q = \frac{1 + r - d}{u - d} = \frac{1 + 0.05 - 0.9}{1.1 - 0.9} = \frac{0.15}{0.20} = 0.75
\]

\[
C_0 = \frac{1}{1 + r} \left( p^{u}C^u + (1 - p^{u})C^d \right) = \frac{1}{1.05} \left( 0.75 \cdot 10 + 0.25 \cdot 0 \right) = 7.14
\]

Pricing options is thus simple in the binomial framework, because the fact that there are only two possible outcomes of the underlying allows us to construct a perfectly risk free portfolio. An alternative derivation would let us construct a perfect replication of a call option using combinations of risk free investments and a position in the underlying.

**A Brain Teaser**

If you have been thinking up to now that all this is straightforward, try to answer the following question.

We have never mentioned anything about the “true” probabilities that prices jump up or down. Moreover, the state-price probability that we have just derived clearly does not depend on the true probabilities. It only depends on the size of the up-jump and the down-jump. The price of the option is thus not determined by the true probabilities. Let us say that the original true probability of an up-jump in the above case is a half. You just had dinner with the CEO of the company whose stock underlies the option, and she revealed to you (after a bottle of nice wine), that next period’s stock price is going to be \( S^u \) with 99% chance, because she is about to announce very favorable news. The rocket scientist in you tells that this should not affect the price of the option. After all, you have just concluded that the option price does not depend on the true probability of an up-jump. Hence, you conclude that the price must remain \( C \). But you may still have a hint of normal human being, and find this conclusion counterintuitive.
Since you know that the stock price is very likely to be $S^u$, you find it more than reasonable that today's option price must be higher, namely close to $C^u$. Which part of your brain is right?

**References**

**Original References**
The basic reference on the binomial option pricing model is Cox, Ross, and Rubinstein (1979).

**Textbook References**
Hull (2006) is a standard textbook.
Problems

14.1 \( ud \) [1]
The current price of the underlying is 50. This price will next period move to either 48 or 60. Find the constants \( u \) and \( d \).

14.2 Call Option [3]
The current price of the underlying is 50. This price will next period move to \( uS \) or \( dS \), where \( u = 1.1 \) or \( d = 0.95 \). If the risk free interest rate is 5%, what is the price of a call option with exercise price 50?

14.3 Call Option [4]
A stock's current price is $160, and there are two possible prices that may occur next period: $150 or $175. The interest rate on risk-free investments is 6% per period.

1. Assume that a (European) call option exists on this stock having on exercise price of $155.
   (a) How could you form a portfolio based on the stock and the call so as to achieve a risk-free hedge?
   (b) Compute the price of the call.

2. Answer the above two questions if the exercise price was $180.

14.4 Calls, Hedge [6]
A stock's current price is $100. There are two possible prices at the end of the year: $150 or $75. A call option to buy one share at $100 at the end of the year sells for $20. Suppose that you are told that

1. writing 3 calls,
2. buying 2 stocks, and
3. borrowing $140

is a perfect hedge portfolio, i.e. a risk free portfolio. What is the risk free rate of interest?

14.5 Call Option [5]
You bought a call contract three weeks ago. The expiry date of the calls is five weeks from today. On that date, the price of the underlying stock will be either 120 or 95. The two states are equally likely to occur. Currently, the stock sells for 96. The exercise price of the call is 112. Each call gives you the right to buy 100 shares at the exercise price. You are able to borrow money at 10% per annum. What is the value of your call contract?
The price of stocks in the "A" company is currently 40. At the end of one month it will be either 42 or 38. The risk free interest rate is 8% per annum. What is the value of a one-month European call option with a strike price of $39?

14.7 Arbitrage [8]
Consider the the binomial option pricing model, where the constants $u$ and $d$ are used to generate future states $S^u = uS$ and $S^d = dS$, and where $r$ is the risk free interest rate. Show that if $d < u < (1 + r)$ an arbitrage opportunity (free lunch) exists.

14.8 MC [4]
MC stock is selling for 30 per share. It is expected that the stock price will be either 25 or 35 in 6 months. Treasury bills that mature in 6 months yield 5%. (p.a.). Use a state-price probability (for the "up" state) of 0.5741.

1. What is the price of an MC option with strike 32?

2. Is the stock priced correctly?
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Chapter 15

Multiple Periods in the Binomial Option Pricing Model

Contents

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As we saw in the previous chapter, the binomial framework allows us to find exactly what the state price probabilities must be to avoid arbitrage. But the setting that has turned the binomial option pricing model into a workhorse of option pricing is one with multiple periods.

15.1 Multiple Periods

The basic assumption is still that every period, the price of the underlying stock can either jump up by a multiplicative factor $u$ or down by a factor $d$. 
But we also assume that this state of affairs persists over time. Each period there is again two branches with jumps defined by the same factors $u$ and $d$:

\[
\begin{align*}
  u(uS) &= uuS \\
  d(uS) &= duS \\
  u(dS) &= udS \\
  d(dS) &= ddS
\end{align*}
\]

Note that since $du = ud$, the four nodes in period two really are only three distinct nodes.
The pricing exercise of the previous chapter can be repeated for every node in a "binomial tree" constructed by replicating the one-period model. To determine the value at a particular point in time, one starts at the end of the tree, where the values of the call option are simply determined as the maximum of the stock price minus strike price, or zero. One then works backward through the tree, determining values as in the previous section.

Let us take the two period example above, where there are three possible stock prices at time 2: \( uuS, udS = duS \) and \( ddS \). The option price changes from \( C_0 \) to either \( C^u_1 \) or \( C^d_1 \) at time 1. Each of these prices is the value of a one period option expiring one period later, at time 2. At time 2 we know the payoffs of the option.

\[
\begin{align*}
C_1^u & \quad \text{max}(S_{u2} - K, 0) \\
C_0 & \quad \text{max}(S_{ud} - K, 0) \\
C_1^d & \quad \text{max}(S_{d2} - K, 0)
\end{align*}
\]

To price we "work backwards." We start with the two possible call values at time 1. These can be calculated using the one period method we used in the previous chapter. Then, given the two possible values at time 1, we are again working with a one period problem, a derivative with two possible payoffs \( C^u_1 \) and \( C^d_1 \) at time 1.

\[
\begin{align*}
C_1^u & \\
C_0 & \\
C_1^d
\end{align*}
\]

This is again solved using the state price probabilities calculated before.
The key assumption here is that the multiplicative movements $u$ and $d$ do not change over time, and we can use the state price probability

$$p^u = \frac{1 + r - u}{u - d}$$

on a period by period basis.

Example
The current stock price $S$ is 100. The multiplicative movements $u = 1.1$ and $d = 0.9$ are constant over time. The per period interest rate $r$ is 5%. The assumptions give the following evolution of the stock price

Let us price a two period European call option with exercise price $K = 95$.

The first step is to find option values at maturity, $C^T = \max(0, S^T - K)$:

$$C^{uu} = \max(121 - 95, 0) = 26$$

$$C^{ud} = \max(99 - 95, 0) = 4$$

$$C^{dd} = \max(81 - 95, 0) = 0$$
Given these payoffs at time 2, we use the state price probability

\[ p^u = \frac{1.05 - 0.9}{1.1 - 0.9} = 0.75 \]

to find time 1 option prices

\[ C^u = \frac{1}{1.05} (0.75 \cdot 26 + 0.25 \cdot 4) = 19.52 \]

\[ C^d = \frac{1}{1.05} (0.75 \cdot 4 + 0.25 \cdot 0) = 2.86 \]

We have now calculated

\[ C^u = 19.52 \]

\[ C^d = 2.86 \]

and we use these period 1 values to find the time 0 call price

\[ C_0 = \frac{1}{1.05} (0.75 \cdot 19.52 + 0.25 \cdot 2.86) = 14.62 \]

The price of the call is 14.62.

15.2 The Binomial Formula and the Black Scholes Model

The pricing formula that this produces when the number of periods increases to say \( n \) is the so-called binomial option pricing model. It would not be appropriate to write down the \( n \)-period formula here. It would be in a specialized options or advanced corporate finance text. For those interested in it, let us merely note that the numbers one produces with the binomial pricing formula are almost identical to those of the famous Black-Scholes model. Of course, this assumes that one picks \( u \) and \( d \) in a way that is related to the stock's volatility (as we did in Chapter 12). The volatility is the single most important parameter in the Black-Scholes formula.
15.3 Early Exercise of Puts in the Binomial Model

With binomial trees, it is easy to determine early exercise. One works backwards through the tree, starting at the end of each branch. At each node, one determines whether it is better to exercise the option (at that point) or keep it alive. If the former is true, one replaces the value of the option (determined by the recursive strategy) with the immediate exercise value. Hence, one effectively “trims” the tree, discarding unnecessary branches.

Example

Use the data about the underlying from the previous example. Find the price of a 2 period American Put option with strike price $K = 100$.

\[
\begin{align*}
P_{uu} &= \max(0, 100 - 121) = 0 \\
P_{ud} &= \max(0, 100 - 99) = 1 \\
P_{dd} &= \max(0, 100 - 81) = 19
\end{align*}
\]

We first calculate the time 1 value of the option if it is not exercised until time 2.

\[
\begin{align*}
P^u &= \frac{1}{1.05} (0.75 \cdot 0 + 0.25 \cdot 1) = 0.24 \\
P^d &= \frac{1}{1.05} (0.75 \cdot 1 + 0.25 \cdot 19) = 5.78
\end{align*}
\]
If the put option is not exercised, we can find the time 0 put price as

\[ P_0 = \frac{1}{1.05} (0.75 \cdot 0.24 + 0.25 \cdot 5.78) = 1.39 \]

This would be the price of an European put option.

But an American put may be optimally exercised early, as we have seen earlier. The possible exercise point is at time 1 when the underlying is at \( S^d = 90 \). Exercising then has a value equal to \( X - S^d = 100 - 90 = 10 \), which is higher than the value of keeping the option "alive," (value of the option if it is kept unexercised till the next period), equal to 5.78.

To price the American option we therefore replace 5.78 with 10 in the calculation of the put price at time 0:

\[ P_0 = \frac{1}{1.05} (0.75 \cdot 0.24 + 0.25 \cdot 10) = 2.38 \]

---

15.4 Adjusting for Dividends in the Binomial Model

Options on a stock paying dividends is another case which is easily solved using the binomial model.

**Example**

We again use the underlying of the previous two examples, but we are now told that the company will pay dividend of $10 next period. If we assume the stock price falls by the amount of the dividend, we get a dividend adjusted binomial tree.
Note that we lost the "linkup" between nodes after the dividend.\footnote{If the dividend had been proportional to the price, that would not have happened. Can you see why?} We want to price a call option with an exercise price $K = 100$. The state price probabilities are the same as the ones we calculated above,

\[ p^u = 0.75 \]

If the option was European we could do nothing next month, we get the following picture of option prices:

But what if the option is American? Then, next period, just before the ex dividend date, we can exercise the option, receiving $C^u = \max(0, S^u - X)$. In the case above, if the stock price goes up to 110 at time 1, we could exercise the option, earning $110 - 100 = 10$. This is better than the value of letting the option "stay alive," which is calculated in the tree above to be 7.14.
The American feature of the call option has value, compare the price of an European call, equal to 5.87, to the price of an American call, 7.875.

15.5 Implementing the Binomial Option Formula

The understanding of the binomial formula is usually much improved by actually implementing it on a computer. To help those of you who are trying to do so, we show an example of how a typical computer algorithm for a binomial price is implemented.

First, a remark on the up and down movements. The standard way of finding these from the volatility $\sigma$ of the underlying asset, the time to maturity $(T - t)$ and the number $n$ of periods in the binomial approximation, is as follows

$$\Delta t = \frac{T - t}{n}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p^u = \frac{e^{r\Delta t} - d}{u - d}$$

A sample implementation in C++ is given below.

The computer algorithm for a binomial above merits some comments. There is only one vector of call prices, and one may think one needs two, one at time $t$ and another at time $t + 1$. (Try to write down the way you would solve it before looking at the given algorithm.) But by using the fact that the branches reconnect, it is possible to get away with the given algorithm, using one less array. You may want to check how this works. It is also a useful way to make sure one understands binomial option pricing.
Figure 15.1 C++ program for an American call option

```c++
#include <cmath> // standard mathematical library
#include <algorithm> // defining the max() operator
#include <vector> // STL vector templates

double option_price_call_american_binomial( double S, // spot price
double X, // exercise price
double r, // interest rate
double sigma, // volatility
double t, // time to maturity
int steps) { // no steps in binomial tree
	double R = exp(r*(t/steps)); // interest rate for each step
double Rinv = 1.0/R; // inverse of interest rate
double u = exp(sigma*sqrt(t/steps)); // up movement
double uu = u*u;
double d = 1.0/u;
double p_up = (R-d)/(u-d);
double p_down = 1.0-p_up;
vector<double> prices(steps+1); // price of underlying
vector<double> call_values(steps+1); // value of corresponding call

prices[0] = S*pow(d, steps); // fill in the endnodes.
for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];
for (int i=0; i<=steps; ++i) call_values[i]=max(0.0,(prices[i]-X)); // payoff at maturity
for (int step=steps-1; step>=0; --step) {
    for (int i=0; i<=step; ++i) {
        call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
        prices[i] = d*prices[i+1];
        call_values[i] = max(call_values[i],prices[i]-X); // check for exercise
    }
}
return call_values[0];
};
```
It is in the case of American options, allowing for the possibility of early exercise, that binomial approximations are useful. At each node we calculate the value of the option as a function of the next periods prices, and then check for the value of exercising the option now.

**References**

Cox, Ross, and Rubinstein (1979) is the general reference on the binomial option pricing model. Good textbook discussions of multiperiod binomial models are Hull (2006) and McDonald (2006). See Ødegaard (2005) for some further discussion of implementation of option pricing formulas.
Problems

15.1 \( ud \) [2]

The current price of the underlying is 100. This price will two periods from now move to either 121, 99 or 81. Find the constants \( u \) and \( d \) by which prices move each period.

15.2 \( Call \) [3]

The current price of the underlying is 50. This price will each period move to \( uS \) or \( dS \), where \( u = 1.1 \) or \( d = 0.95 \). If the per period risk free interest rate is 5\%, what is the price of a two period call option with exercise price 50?

15.3 \( HAL \) [6]

You are interested in the computer company HAL computers. Its stock is currently priced at 9000. The stock price is expected to either go up by 25\% or down by 20\% each six months. The annual risk free interest rate is 20\%.

Your broker now calls you with an interesting offer.

You pay \( C_0 \) now for the following opportunity: In month 6 you can choose whether or not to buy a call option on HAL computers with 6 months maturity (i.e. expiry is 12 months from now). This option has an exercise price of $9000, and costs $1,500. (You have an option on an option.)

1. If \( C_0 \) is the fair price for this "compound option," find \( C_0 \).

2. If you do not have any choice after 6 months, you have to buy the option, what is then the value of the contract?

15.4 \( HS \) [4]

The current price of a stock in the "Hello Sailor" entertainment company (HS) is $100. Each period the stock price either moves up by a factor \( u = 1.5 \) or down by a factor \( d = 1/u \). (HS is in a highly variable industry.) The per period interest rate is 5\%.

Consider the following "compound option" written on HS: You buy the option at time 0. The compound option gives you the right to at time 1 choose to receive one of the following two options:

- A call option on HS with exercise price 100 and expiring in period 2.
- A put option on HS with exercise price 100 and expiring in period 2.

Calculate the price of this "compound option."
Chapter 16

An Application: Pricing Corporate Bonds

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16.1 Corporate Bonds

We now apply our methods for valuing risky payoffs to valuing corporate debt. When a corporation issues a bond, it issues a security which promises fixed future payments. We have already seen how to price such bonds in the risk free case.

\[ B = \sum_t P_t X_t, \]

where \( P_t \) is the current price of getting one dollar for sure at time \( t \), and \( X_t \) is the bond payment in period \( t \). But this framework cannot in general be applied to corporate bonds, because these bonds are risky. The CAPM type of framework is also problematic for risky bonds, due to the contingent nature of payoffs. The right valuation framework is therefore the state-price approach.
16.2 The Risky Part of Corporate Bonds

Corporate bonds are risky because of the default probability. To illustrate how this affects values consider the following example.

Example
The Are You Lucky Gold Mine Company has only two mines left. One mine has well known reserves, they are estimated to yielding a net cashflow of 1 million next period for sure, and nothing afterwards. The other mine is not yet fully explored. The cost of exploring enough to estimate the reserves is 100,000. There is a large probability the mine is empty, in which case it is worthless. There is a small probability of large gold reserves in the mine, supporting an annual production with net cashflows of 1 million per year for 5 years. The Are You Lucky Gold Mine Company has issued debt with a face value $D$ of 1,500,000. The debt is due next period.

It is only in the case that the new mine is nonempty that the gold company has sufficient cash flow to pay back the debt of 1.5 million. Otherwise the firm only has assets worth 900,000 to cover 1.5 million of debt. The latter case is default, the firm goes bankrupt and its assets are taken over by the creditors, in this case the bondholders.

It is the nontrivial probability that a company goes bankrupt that makes corporate bonds risky. Since these bonds are risky, they should be valued using the by now familiar state price approach.

Example
What is the value to bondholders of the debt of the “Are You Lucky Gold Mine” company?

The relevant “states” are here defined by the value of the company, termed $V$:

$V^u$ (Find gold in second mine)

$V$ (Do not find gold)

What are the payments to bondholders ($B$)? In the “up” state the assets of the firm is worth more than the face value of debt, bondholders receive the face value. In the second case the firm is bankrupt, with total assets of 900,000 (The one million from the old mine less the cost of 100,000 of exploring the new mine).
16.3 Risk Free Bonds with Put Options

To price we need the state price probability $p^u$. This could for example be estimated from the stock price of the company. Supposing $p^u = 0.48$ and $r = 0.05$, we can value the bonds of the company as

\[
B = \frac{1}{1 + r} \left( p^u B^u + (1 - p^u) B^d \right) = \frac{1}{1 + 0.05} (0.48 \cdot 1.5 + (1 - 0.48)0.9) = 1,131,000
\]

16.3 Corporate Bonds are Risk Free Bonds with a Put Option Attached

To build further intuition about bond pricing, we can reinterpret the position for a bondholder as holding a risk free bond, but having sold a put option with exercise price equal to the face value $D$ of the debt.

Let $V'$ be next period's value of the firm. $E$ is today's value of equity. A company will default if it does not have enough assets to pay the debtholders. Hence, the debtholders receive the following payoff pattern:

\[
\min(V', D),
\]

which can be re-written more lucidly as:

\[
D - \max(D - V', 0).
\]

This means that corporate debt is really a packet of

1. riskfree debt with face value $D$.

2. a put option that the debtholders wrote to the shareholders.

By the axiom of value additivity, we can therefore value corporate debt as the sum of the value of riskfree debt minus the value of a put option.

\[
B = \frac{1}{1 + r} D - \frac{1}{1 + r} E^*[\max(D - V', 0)].
\]  \hspace{1cm} (16.1)
Figure 16.1 Bond as combination of risk free bond and a short put.

Since equity is the residual claimant, by value additivity the value of the firm \( V \) has to equal the sum of equity and debt, \( V = E + B \). That means we can also calculate the value of corporate debt from the value of equity \( E \):

\[
B = V - E.
\]

We already discussed that \( E \) is the value of a call option on the assets of the firm, with payoff pattern

\[
\max(V' - D, 0).
\]

Corporate debt can therefore also be considered a combination of the assets of the firm minus a call option written to the shareholders. The valuation formula would be:

\[
B = V - \frac{1}{1+r} E^* \left[ \max(V' - D, 0) \right] \quad (16.2)
\]

The combination is illustrated in figure 16.2.

All of the approaches to pricing debt above are consistent, and either approach will give the same numerical result.

Example

A company is set up to implement a project that requires an $80 investment outlay. The present value of the project is $90, giving a NPV of \( 90 - 80 = 10 \). The future value \( V_1 \) of the project is one of two: \( V^u = 110 \) (with probability 0.8) or \( V^d = 80 \) (with probability 0.2).
Note that these are the true probabilities, not the state price probabilities. We can use them to calculate the expected return of the project to be

\[ E[r] = \frac{0.8 \cdot 110 + 0.2 \cdot 80}{80} = \frac{104}{80} = 30\%. \]

To finance the project, the management of the company decides to issue debt with a face value of $85. The riskfree rate is 5%. We will first find how much the debt can be issued for.

To value the debt we will use relationship (16.1) above. To do so we need state price probabilities. These we can find from the value of the project, since the entire firm is also valued using the same state price probabilities:

\[ V = \frac{1}{1 + r} E^*[V_t] = \frac{1}{1 + r} (p^u V^u + (1 - p^u) V^d) \]

Plugging in numbers

\[ 90 = \frac{1}{1.05} (p^u 110 + (1 - p^u) 80) \]
and solving for \( p^u \) we find \( p^u = 0.48 \). Using these state price probabilities we can find the current value of debt as

\[
B = \frac{1}{1+r}D - \frac{1}{1+r}E^*[\max(D - V_1, 0)]
\]

\[
= \frac{1}{1.05} \times 85 - \frac{1}{1.05} \times (0.48 \times 0 + (1 - 0.48) \times (85 - 80))
\]

\[
= 78.5.
\]

The corporate debt with a face value of 85 is issued for 78.5 one period before. We can use this to find the return from buying the bond, or the yield on the corporate bond, as

\[
\frac{D}{B} - 1 = \frac{85}{78.5} - 1 = 8.3%.
\]

The return from buying the risky debt is 3.3% higher than the 5% risk free interest rate. This reflects a risk premium necessary to induce anybody to be willing to buy the debt. Let us also value the debt using the alternative approach shown in (16.2) above:

\[
B = V - \frac{1}{1+r}E^*[\max(V_1 - D, 0)]
\]

\[
= 90 - \frac{1}{1+r} \times (p^u \max(110 - 85, 0) + (1 - p^u) \max(80 - 85, 0))
\]

\[
= 78.5.
\]

We can use the relation between debt and equity to find the value of equity

\[
B = V - E
\]

giving an equity value of

\[
E = V - B = 90 - 78.5 = 11.5.
\]

To get the company going, the shareholders will have to pay the remainder of the investment outlay, 80 – 78.5 = 1.5. For 1.5, they get shares that are worth 11.5. The difference of 10 is of course the NPV of the project.

### 16.4 The Incentives for Shareholders to Change the Nature of the Assets Once the Bonds are in Place

We will now show that corporate finance is not so simple as you thought, by illustrating that the incentives of different groups, namely bondholders and shareholders of a given firm, may not be aligned.

**Example**

Suppose the management in the previous example went ahead with the 85 debt issue, yielding 78.5. In addition the firm issued shares for 1.5. Just before starting the $80 project, management notices that a nice alternative project presents itself. The alternative project also costs 80, has the same \( NPV = 10 \), but the following payoff pattern:
16.4 Shareholder Incentives

Using the true probabilities we can calculate its expected return as

$$E[r] = \frac{0.5 \cdot 150 + 0.5 \cdot 40}{80} = 18.75\%$$

Note that the new project has a lower expected return than the alternative one. If you were to calculate the volatility, or standard deviation, of the return on the two projects, you would find that the alternative project has a higher standard deviation, it is more variable.

Should management switch projects? If the management wants to pick the project that maximizes the value of the firm, it would be indifferent, since both projects have the same value. If the management were (erroneously) trying to maximize expected return on value, it should stick to the old project. But managers are responsible to shareholders, so they should pick whatever project gives the highest share value. What is the share value under the new project?

We first use the NPV of the project to determine the state price probabilities under the new project, denoted $\tilde{p}^u$

$$V = 90 = \frac{1}{1.05} (\tilde{p}^u150 + (1 - \tilde{p}^u)40)$$

giving a state price probability $\tilde{p}^u = 0.50$. Use $\tilde{p}^u$ to determine the value of equity:

$$B = \frac{1}{1 + r} (\tilde{p}^u(150 - 85) + (1 - \tilde{p}^u)0) = \frac{1}{1.05} \cdot 0.50 \cdot 65 = 31$$

The share value under the new project is 31. Comparing this to the share value under the original project of 11.5, the management should clearly switch to the new, more risky project.

Switching projects really amounts to “expropriating” the bondholders. If the bondholders had known the company would use the raised capital to invest in the alternative, more risky project, they would only have been willing to put up

$$B = V - E = 90 - 31 = 59$$

to finance the project. That is, the bondholders would demand an increase in the yield of the bond to

$$\frac{D}{B} - 1 = \frac{85}{59} - 1 = 44\%$$

to compensate for the increased risk of default. But since the bonds have already been issued at 78.5, management has the cash on hand... So, yes, the management should switch projects, and the debtholders bear the cost of the increased riskiness.
In fact, this is a fairly general result. After the debt is in place, management always has an incentive to increase the riskiness of the firm, thereby increasing the value of equity, while reducing the value of debt.

*Debt covenants* are meant to keep the management from doing so. Or, alternatively, the management could attach *warrants* to the bonds (see Chapter 13), or issue *convertible bonds*. Let us discuss the latter.

### 16.5 A Solution: Convertible Bonds

A convertible bond gives the holder the right to exchange the bond for a number of shares of equity. As such they combine features of bonds with features of warrants.

**Example**

A corporation has issued convertible bonds with a face value of 1000. At the discretion of the bondholder each bond can be exchanged for \(33\frac{1}{3}\) shares of newly issued common stock.

The *conversion ratio* of a convertible bond is the specification of how many shares are issued per bond. The *conversion price* is the price of the underlying stock which makes the face value of the convertible bond equal to the value of the converted bond.

**Example**

In the previous example the conversion ratio is \(1 : 33\frac{1}{3}\) and the conversion price is 30.

Convertible bonds create problems that are similar to those for warrants. "Dilution" is a confusing concept, and optimal conversion policies are difficult to determine. We will avoid problems created by conversion and refer a study of them to more advanced texts. Let us look at an example where there is *one* share and *one* convertible bond, and the conversion ratio is 1:1. The notation is the same as earlier in this chapter.

Then, the terminal payoff on the convertible is as follows.

1. If \(D < V' < 2D\), then the payoff is \(D\).
2. If \(V' \leq D\), then the company defaults and the payoff is \(V'\).
3. If \(2D \leq V'\) then it pays to convert the bond and receive the value of one share, which is \(V'/2\).

Graphically, the payoff pattern is shown as in figure 16.3.

A convertible bond can be valued as the combination of

1. a risk free bond.
2. (writing) a put option.

3. a call option on the shares of the firm.

Figure 16.4 illustrates the combinations

With value additivity, this gives the following valuation formula:

\[
B = \frac{1}{1+r} D - \frac{1}{1+r} E^* \max(D - V', 0) + \frac{1}{1+r} E^* \max\left(\frac{V'}{2} - D, 0\right).
\]
16.6 Callable Convertible Bonds

Many firms attach a call provision to convertible bonds. That is, the firm reserves the right to call back the bond issue at a pre-set call price. When called, bondholders do have the right to convert the bonds to equity instead of delivering the bonds against the call price. Let $\bar{C}$ denote the call price. An interesting question is: when should the management call the convertible bonds? The answer is that whatever policy the management uses, it should be in the interest of the shareholders. Since

$$E = V - B$$

and (unlike warrants) the value of the firm is unaffected by conversion, $E$ is maximized by minimizing the value of the convertible bonds. That will happen if the management chooses the conversion policy that minimizes the value of the conversion option. How? By calling as soon as the conversion option is in the money, that is, the conversion value rises above the call price. The result of the call is obviously that the bondholders will convert, because the conversion value is at least as large as the call price. The mechanics and the effect on valuation are best understood with an example.

Example

We look at a situation with three dates, $t = 0$, $t = 1$ and $t = 2$. The riskfree rate per period is 10%. The evolution of the firm’s value at $t$, $V_t$, can be described by a binomial tree:

These numbers were chosen such that the (per period) state-price probability is always the same

$$p^u = 0.40.$$
There is one share of equity and one convertible bond. The bond has a face value of $D = 36$, to be paid at $t = 2$. There is no intermediate coupon payment. The bond is callable only at $t = 1$ for $\bar{C} = 37$. The conversion ratio is 1:1.

Ignoring the call provision, the bond can be valued by recursively applying our valuation formula (10.1) along the branches of the binomial tree. The result is:

Now consider the call feature. It is optimal to call when the continuation value (of the bond) is more than the call price $\bar{C} = $37. This is the case in the "up"-state at $t = 1$. Notice, however, what happens. In that state, the conversion value is $37.3$ (total firm value $74.6$, divided by the two shares that will exist after conversion). So, rather than tendering their holdings to the firm and get only $37$, the bondholders will convert and receive a share worth $37.3$. That is, the call effectively forces conversion. In the "down"-state, nothing changes, because the continuation value of the bond is below the call price. The resulting effect on today's value of the convertible bond is:

Note that the continuation of the tree in the "up"-state at $t = 1$ is cut off, the tree is "trimmed" because of the firm's exercise policy.
References

See Smith (1976), Smith (1979) and Cox and Rubinstein (1985) on convertible bonds. See Myers (1977) on investment distortions. See also the recent summary by Myers (2001).
Problems

16.1 *Conversion* [1]
Why does conversion of convertible bonds not affect the value of the firm?

16.2 *Bond Covenants* [3]
In one or two sentences, answer the following.

1. Who benefits from the covenants in bond contracts when the firm is in financial trouble? Why?

2. Who benefits from the covenants in bond contracts when the firm is issuing debt? Why?

16.3 [3]
The Q corporation will next period realize a project that will have value either 100 or 20. This project is the only assets that Q corporation have. Q has issued a bond with face value of 50, due next period. The risk free interest rate is 10% and the current value of equity in Q is 40. Determine the current value of the bond.

16.4 *Projects* [7]
A start-up company considers two investment projects that require a $90 investment. Both are zero NPV projects (hence, the value of the project is $90). Only one project can be implemented. Part of the necessary funds are to be acquired through a zero-coupon debt issue, with face value of $85. The remainder is collected through an equity issue. The specifics of the two projects are

- **Project A:**
  
  \[
  \text{Final value} = \begin{cases} 
  110 & \text{with probability 0.8} \\
  80 & \text{with probability 0.2} 
  \end{cases}
  \]

- **Project B:**
  
  \[
  \text{Final value} = \begin{cases} 
  150 & \text{with probability 0.5} \\
  40 & \text{with probability 0.5} 
  \end{cases}
  \]

1. Which project has the highest expected return?

2. Assume management randomly picks a project, and decides to choose Project A. (Management should not have any particular preference, because both projects have a zero NPV.) How much money will the company raise from the debt issue? How much equity will be raised? Assume that the risk free interest rate equals 5%.
3. Assume now that the debt and equity issues to finance project A have been completed. Now the management turns around and does not implement project A, but, instead, goes ahead with project B. What is the loss in value to the debtholders? Would the equityholders applaud this move?

16.5 Convertible [6]
Suppose the firms end of period value will be:

$$\text{Value} = \begin{cases} 1500 & \text{with probability 0.6} \\ 800 & \text{with probability 0.4} \end{cases}$$

Today's firm value is 1000. The risk free rate is 5%. The firm has 10 shares of equity and 100 convertible bonds with a face value of 10 each. The bond pays no coupon. One bond can be converted into one share.
Compute the value of the convertible bond and of the equity.

16.6 Yazee [4]
Yazee is valued at $V = 100$. Tomorrow's value $V'$, will be either 150 or 50, with equal chance, Yazee has issued a corporate bond with face value 100 and no coupon, to be paid tomorrow. The bonds presently give a 15% yield. Is the bond mispriced? The risk free rate is 10%.

16.7 AoB [4]
AoB, Inc., has issued 2 shares of common stock and 1 convertible bond. AoB is valued at $V = 100$. Tomorrows value, $V'$, will be either 150 or 50, with equal chance. The bond has a face value of $30 and carries a coupon of 10%. The conversion ratio is 1:1. The bond is due tomorrow. The risk free rate is 10%. Value the bond.
Part IV

Corporate Finance
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Chapter 17

Are Capital Structure Decisions Relevant?

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17.1 The Capital Structure Problem

Remember our definition of $V$:

The value of a firm is the price for which one could sell the stream of cash flows that the assets of the firm generates for the traditional creditors.

Because of value additivity and no free lunches, $V$ equals the market value of the liabilities to the traditional creditors (see Chapter 2). Usually, we distinguish only equity, with a value $E$, and debt, with a value $B$. Hence,

$$ V = E + B $$

Note we have not defined $V$ to be equal to $E$ plus $B$. We wanted to give $V$ a separate life, because this allowed us to determine $V$ separately, and to derive $E$ and $B$ from $V$ using derivatives analysis.
Until now, we have taken the value of the firm to be exogenous. We derived the value of equity and debt from that of the firm. The capital structure problem concerns whether there could be a feedback effect, or, could changing the capital structure (debt/equity mix) change the value of the firm? This is the capital structure problem:

- Does the way a firm is financed (its debt/equity mix) affect the firm value?
- If it does, what is the optimal debt/equity mix, the one that maximizes the value of the firm?

In this and the next chapter we will figure out whether we are really justified in ignoring any feedback effects. As it turns out, under fairly general conditions there is no feedback, firm value is independent of the capital structure. Showing this result and under what conditions it holds is the topic of this chapter. In the next chapter we will look at conditions that make that intuition break down. In particular we look at how taxes affect things.

17.2 The Capital Structure Problem when Assets are In Place: Modigliani Miller I

We will provide a clear answer to the capital structure problem under the assumption that the company's assets are already "in place". By that we mean that investment took place and is irreversible. It can then be shown that the value of the firm does not depend on the debt/equity mix. Showing this is one of the most famous results in finance, the Modigliani Miller Theorem I (MM I).

The main result can be stated as

If we assume that the change in financing mix does not affect the total cash flows of the firm accruing to the shareholders and bondholders, changing the debt/equity ratio does not change \( V \), the value of the firm.

Intuitively, it can't possibly affect \( V \), because "same cash flows implies same value." Otherwise there would be an arbitrage opportunity (free lunch). In their original proof of the above proposition, Modigliani and Miller exploited the arbitrage opportunity in an ingenious way. We will go through their argument. Assume there are two firms, a levered one (one with both debt and equity), with value \( V_L \), and an unlevered one (one with only equity), with value \( V_U \). Debt is perpetual, paying a coupon \( C \) per period. All cashflows are risk free, hence we
can discount future coupon payments $C$ at the risk free rate $r$ to find the value of debt $B_L$:

$$B_L = \frac{C}{r}.$$ 

Now let $E_L$ and $E_U$ denote the value of equity in the levered and unlevered firm, respectively. Of course, $E_U = V_U$. For simplicity, suppose both firms earn a perpetual stream of $F$ dollars per period. The levered firm pays $C$ out of $F$ and passes the remainder, $F - C$, to the shareholders. The unlevered firm passes on the entire flow $F$ to shareholders. If the debt equity mix does not affect firm value, $V_U = V_L$. The MM proof consists of showing that if this does not hold, an arbitrage opportunity will exist.

Take the case where $V_U < V_L$. An arbitrageur could buy, say, 15% of the equity of the unlevered firm. This will generate a cash flow of $0.15F$ per period. To finance this purchase, the arbitrageur sells short 15% of the equity of the levered firm, that is, he promises to deliver

$$0.15(F - C)$$

in perpetuity. That leaves him with $0.15C$ per period. The arbitrageur borrows in perpetuity as much as he can cover with an interest payment of $0.15C$. The bank will lend him:

$$\frac{0.15 \cdot C}{r} = 0.15B_L.$$ 

So, our arbitrageur has no net cash in/outflow in the future. At present, however, he has a cash inflow equal to:

$$-0.15E_U + 0.15E_L + 0.15B_L = 0.15(V_L - V_U) > 0.$$ 

Our arbitrageur can go out and have a free lunch. We leave it to the reader to argue for the existence of an arbitrage opportunity if $V_U > V_L$.

We will look at an example where a firm decides on how large a bank loan to take out to finance a project. Unlike in the MM I proof, the (bank) debt in the example is risky. That does not affect the MM I proposition, of course. We will investigate how the bank changes the interest rate it charges on the loan as the size of the loan increases.

Example

Mobell wants to build a jet fuel tank at the Jackson Hole (Wyoming) airport. The tank is to be owned and operated by a wholly-owned subsidiary, JackMo. Building the tank costs 5. The tank holds 1,000 liter, and the current price of one liter is 0.40. Let us consider two periods only. Today, the tank is built and filled. Next period JackMo sells the entire content of the tank. Handling and delivery costs are 0.10 per liter. The price at which JackMo will
Are Capital Structure Decisions Relevant?

be able to sell the jet fuel is either 0.80 per liter or 0.50 per liter. The riskfree interest rate is 10%. Today, the futures quote for a contract to deliver one liter of jet fuel tomorrow is 0.55.

Mobell decides to finance JackMo partly through a bank loan. It approaches First Yellow Bank. It asks for a interest rate quote for loans with face values of 200 and 400 dollars. The remainder will be financed through an equity issue to Mobell.

What is the rate that First Yellow Bank should charge on those loans? Does the value of JackMo remain the same, whether the 200 loan or the 400 loan is chosen? How does the average rate of return on equity change with increased leverage?

Let us first find the cash flows for JackMo. The investment is $5 + 0.4 \cdot 1000 = 405$. The period 1 cash flow depend on the oil price at time 1. If we let $S$ be this oil price, the cash flow is $1000 \cdot (S - 0.10)$. The possible cash flows for JackMo is thus

To price JackMo we need state-price probabilities for the two possible "oil price states." These can actually be found from the futures price. We will return to pricing of futures in chapter 21, but for now the important thing about them is that futures prices are set so that the NPV of investing in a futures is zero. This can be used to find the state price probability

$$ 0 = \frac{1}{1.10} (p_u (0.80 - 0.55) + (1 - p_u)(0.5 - 0.55)) $$

Solving for $p_u$ we find

$$ p_u = 0.167 $$

The current value of JackMo is then calculated as

$$ \text{value} = \frac{1}{1.1} \left( 0.167 \cdot 700 + (1 - 0.167)400 \right) \approx 409 $$

Let us see how this value is distributed on debt and equity, first in the case of the bank loan of 200. Let $D^{200}$ be the face value of the debt. For the bank, lending money to JackMo should be a zero-NPV proposition. JackMo always have enough cash flow to pay

$$ 200 = \frac{1}{1 + r} \left( p_u \min(D^{200}, 700) + (1 - p_u) \min(D^{200}, 400) \right) $$

$$ 200 = \frac{1}{1 + r} \left( p_u D^{200} + (1 - p_u)D^{200} \right) $$

$$ 200 = \frac{D^{200}}{1 + r} $$
\[ D^{200} = 200 \cdot 1.1 = 220 \]

Alternatively one can argue: The loan of 200 is risk free, the bank will charge the risk free rate of 10% on it.

\[ E = \frac{1}{1.1} \left( p^u (700 - 220) + (1 - p^u)(400 - 220) \right) \]
\[ = \frac{1}{1.1} \left( p^u 480 + (1 - p^u) 180 \right) \approx 209 \]

The sum of debt and equity is 200 + 209 = 409, the same as the all-equity financed firm above.

In the case of the 400 loan, let \( D^{400} \) be the sum of face value and interest.

\[ 400 = \frac{1}{1.1} \left( p^u \min(D^{400}, 700) + (1 - p^u) \min(D^{400}, 400) \right) \]
\[ = \frac{1}{1.1} \left( p^u D^{400} + (1 - p^u) 400 \right) \]

\[ D^{400} = 640 \]

Note that JackMo will default in the down state.

The bank is charging an interest rate of \( \frac{640}{400} - 1 = 60\% \). The value of equity in this case is

\[ E = \frac{1}{1.1} \left( p^u (700 - 640) + (1 - p^u) 0 \right) \approx 9 \]

The sum of debt and equity is 400 + 9 = 409. The value of the firm is the same, no matter what the debt/equity mix is. This is MM I. The value of the firm is independent of the loan amount, because the loan does not affect the cash flows in either state.

17.3 Is Maximizing the Value of the Firm Optimal for Shareholders?

Why did we cast the problem of this chapter in terms of maximizing the value of the firm as opposed to the value of shareholding? After all, management is accountable to shareholders (only)! In fact, we should ask: when is maximizing the value of the firm identical to maximizing the value of equity?

Since we analyzed valuation using option pricing theory, the answer is actually very simple. Equity should be considered a call option on the assets of the firm. In particular, if \( V' \) is the future value of the firm, and \( D \) denotes the face value of debt plus interest payments, equity pays: \( \max(V' - D, 0) \). From option pricing theory, we know that the value of a call option increases with the value of the underlying asset, which in the case of equity is \( V \). Thus, increasing \( V \) increases
Are Capital Structure Decisions Relevant?

$E$, maximizing the value of the firm is consistent with maximizing the value of equity.

But some care should be used in interpreting this result. The value of a call option increases with the value of the underlying asset when everything else is kept constant. Other features of the option, like the volatility of the underlying, the interest rates, and the exercise price ($D$ in the case of equity) must be kept constant.

### 17.4 Implications for Cost of Equity Capital: MM II

Since $V$ remains fixed, no matter what the equity/debt financing mix is, we can derive the second Modigliani-Miller proposition. It expresses the (required) rate of return on equity as a function of that on the assets of the firm and the firm’s debt. Of course, the expressions are valid only for the special world that Modigliani and Miller live in (riskfree cash flows, perpetual payments, etc.).

Let $r_V$, $r_E$ and $r_B$ denote the rates of return on the assets of the firm, equity and debt, respectively. Then:

$$r_E = r_V + \frac{B_L}{E_L}(r_V - r_B).$$  \hspace{1cm} (17.1)

In fact, this is just the implication of value additivity that we have been using throughout. It can be derived from looking at the value of equity as the difference of the value of the firm and its debt.

$$E_L = V_L - B_L.$$  \hspace{1cm} (17.1) is then obtained using $r_V = \frac{E}{V}$, $r_B = \frac{C}{B_L}$ and $r_E = \frac{E - C}{E_L}$.

Analogously, the required rate of return on equity is determined by that on the assets of the firm and that of the firm’s debt.

Example

Let us continue the JackMo example, and now assume that the probability of the “up” state is 20%. We can compute the expected return on equity for the two loans:

$$E[r_E^{200}] = \frac{0.2(700 - 220) + 0.8(400 - 220)}{209} = 15\%$$

$$E[r_E^{400}] = \frac{0.2(700 - 640) + 0.8(400 - 400)}{9} = 33\%$$

Figure 17.1 illustrates MM II (for a risky world). The required rate of return on equity increases with the amount of debt in the capital structure.

Let us also calculate the returns on the unlevered firm and the debt

$$E[r_V] = \frac{0.2 \cdot 700 + 0.8 \cdot 400}{409} = 12.5\%$$
Figure 17.1 MM II for JackMo

![Graph showing expected return on debt for different values of debt to equity ratio.]

\[ E[r_E^{200}] = r_f = 10\% \]
\[ E[r_E^{400}] = \frac{0.2 \cdot 640 + 0.8 \cdot 400}{400} = 12\% \]

The expected return on debt also increases to reflect its risk.

Just an aside: Would First Yellow Bank be willing to provide any size bank loan? You’ll observe that there are loan amounts (less than $405) such that First Yellow Bank cannot charge an interest high enough that it’s worth for them to provide a loan. Hence, JackMo is credit-constrained. The credit constraint is a very natural consequence of the design of bank loans. Notice also that, as the state-price probability of the “default” state increases (perhaps because the actual chances of JackMo defaulting increases), First Yellow Bank will lower the amount it can possibly lend. Hence, JackMo will be further constrained.

A remark here, though. There is something deeply unsatisfactory about (17.1). Because everything is riskfree is Modigliani and Miller’s world, \( r_V = r_B = r_f \), where \( r_f \) is the risk free interest rate. Hence, \( r_E = r \), and does not change with leverage, unlike what (17.1) seems to indicate! To really get an idea about how the required rate of return on equity goes up with leverage, one must introduce risk. Since we know how to price equity and debt in a risky world, we are able to come up with clean answers. In particular, the JackMo example illustrates that both Modigliani and Miller’s theory continues to hold in a risky world, the required rate of return goes up with leverage. A picture like figure 17.2 is closer to the true situation.

This should also convince the skeptics among you, who may still not be sure what all this derivatives analysis is good for. The previous remark applies, mutatis mutandis, to all other Modigliani-Miller cases which we will discuss.
17.5 What if Assets are not In Place?

The assumption that "Assets are in place" is very important.

Example
Recall the example from section 16.4, where we saw that we were able to make the shareholders better off by switching to a riskier project after the financing was in. In that example, the value of the firm $V$ did not change upon the switch, but the value of equity $E$ (per share or in toto) definitely increased! We know why switching to a riskier project is advantageous to shareholders. Equity is a call option in the company, and the value of a call increases with the volatility of the underlying asset, keeping everything else constant. Note that we kept the value constant across projects in that example.

If we allow the "assets in place" to change, and the firm management maximizes the value of the shareholders equity, it may even happen that an increase in the debt ratio will make the management take on very risky negative NPV projects in order to increase the value of equity.

References

Modigliani and Miller (1958), Modigliani and Miller (1963), Modigliani and Miller (1969) are the original references for the Modigliani–Miller theorems. Assets in place changes are discussed in Myers (1977). For some recent views on the state of the Modigliani Miller results see Miller (1988), Modigliani (1988), Ross (1988),
Problems

17.1 Debt/Equity [7]
Firm Z and Y have identical cash flows. Firm Z is 40% debt financed and 60% equity financed, while firm Y is 100% equity financed. The same required rate of return on their debt equals 10%. (Assume debt is perpetual)

1. Next period’s cash flows for each firm are $100. Assume both firms pay out all excess cash in the form of dividends. What cash flows go to the debt and equity holders of both firms? Assume no corporate taxes. (Use $D_z$ for the value of firm Z’s debt).

2. You own 10% of firm Z’s stock. What cash flow will you get in the future? What combination of other assets will give you the same cash flow?

3. Suppose the value of firm Z is greater than firm Y. How can you become very rich? (You may assume no transactions costs, or other market imperfections)

4. Now, suppose there is a corporate tax rate of 40%. What should the value of each firm be?

17.2 Frisky [4]
Frisky, Inc is financed entirely by common stock which is priced according to a 15% expected return. If the company re-purchases 25% of the common stock and substitutes an equal value of debt, yielding 6%, what is the expected return on the common stock after the re-financing?

17.3 JB [4]
JB Manufacturing is currently an all-equity firm. The equity of firm is worth $2 million. The cost of that equity is 18%. JB pays no taxes. JB plans to issue $400,000 in debt and use the proceeds to repurchase equity. The cost of debt is 10%.

1. After the repurchase the stock, what will the overall cost of capital be?

2. After the repurchase, what will the cost of equity be?

17.4 LRC [3]
You invest $100,000 in the Liana Rope Company. To make the investment, you borrowed $75,000 from a friend at a cost of 10%. You expect your equity investment to return 20%. There are no taxes. What would your return be if you did not use leverage?
17.5 OFC [5]
Old Fashion Corp. is an all-equity firm famous for its antique furniture business. If the firm uses 36% leverage through issuance of long-term debt, the CFO predicts that there is a 20% chance that the ROE (Return on Equity) will be 10%, 40% chance that the ROE will be 15%, and 40% chance that the ROE will be 20%. The firm is tax-exempt. Explain whether the firm should change its capital structure if the forecast of the CFO changes to 30%, 50% and 20% chances respective for the three ROE possibilities. That is, tell us whether the value of assets and equity change as a result of the changes in ROEs.

17.6 V&M [5]
Note: In the question you are asked to assume risk neutrality. This means that the state price probabilities are not colored by risk aversion (fear) so they are equal to the estimated probabilities in the question.
VanSant Corporation and Matta, Inc., are identical firms except that Matta, Inc., is more levered than VanSant. The companies’ economists agree that the probability of a recession next year is 20% and the probability of a continuation of the current expansion is 80%. If the expansion continues, each firm will have EBIT of 2 million. If a recession occurs, each firm will have EBIT of 0.8 million. VanSant’s debt obligation required the firm to make 750,000 in payments. Because Matta carries more debt, its debt payment obligations are 1 million. Note: EBIT is short for Earnings Before Interest and Taxes. Often this is considered a good estimate of cash flows before interest and tax payments.
Assume that the investors in these firms are risk-neutral and that they discount the firms’ cash flows at 15%. Assume a one-period example. Also assume there are no taxes.

1. Duane, the president of VanSant, commented to Matta’s president, Deb, that his firm has a higher value than Matta, Inc, because VanSant has less debt and, therefore, less bankruptcy risk. Is Duane correct?

2. Using the data of the two firms, prove your answer.

3. What might cause the firms to be valued differently?

17.7 Negative NPV? [3]
Do you agree or disagree with the following statement? Explain your answer.

A firm’s stockholders would never want the firm to invest in projects with negative NPV.
Chapter 18

Maybe Capital Structure Affects Firm Value After All?

Contents

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In the previous we saw that the financing mix was irrelevant for firm value. But maybe some of the conditions there does not hold in practice? Maybe the financing mix does change the value of the firm?

18.1 Only Through Changes in Assets

The financing mix affects the value of the firm only through effects on available cash flows from assets. Under the axioms of corporate finance, changes in the financing mix can affect the value of the firm $V$ only if it lowers or raises the cash flow from the firm's assets that accrue to the traditional creditors.

Example

In a (hypothetical) country named Fundasio, the borrowing of money at a fixed, positive interest rate is taxed with a punitive levy of 5 centimes per diram borrowed. Equity-like contracts remain untaxed. In such a case, debt lowers the cash flow available to the traditional creditors by 5 centimes a diram of debt. The value of the firm is reduced commensurately. It
is optimal for the creditors never to take out any interest-bearing loans. Equity-like contracts such as warrants are preferred.

18.2 Corporate Taxes

In the United States and some other countries, debt carries a distinctive tax advantage for the issuer, coupon payments are deductible from taxable earnings. It is therefore in the interest of corporations to increase debt obligations in order to minimize tax payments. Minimizing tax payments will maximize the cash flow from the assets of the firm that is available for distribution to the traditional creditors. In other words, debt enhances the value of the firm.

Here is the argument that Modigliani and Miller used in the follow-up to MM I and MM II. The assets in place of two firms generate a perpetual, riskfree stream of $F$ dollars per period. The earnings of both firms are taxed at a corporate rate equal to $\tau_c$ percent. At an interest rate of $r$ percent, the value of the unlevered firm, $V_U$, equals:

$$V_U = \frac{F(1 - \tau_c)}{r}.$$

The value of the levered firm equals:

$$V_L = \frac{F - (F - C)\tau_c}{r} = \frac{F(1 - \tau_c) + C\tau_c}{r} = V_U + \tau_c B_L.$$

From this, it is clear that the value of the firm is maximized by taking out as much as debt as possible!

As before, use value additivity to get an expression for the (required) rate of return on the levered firm's equity ($r_E$) as a function of the rate of return on the assets of the unlevered firm ($r_{V_U}$) and that on debt ($r_B$). Value additivity implies:

$$V_L = E_L + B_L.$$

Hence:

$$r_E = \frac{E_L + B_L(1 - \tau_c)}{E_L} r_{V_U} - \frac{B_L}{E_L} r_B (1 - \tau_c).$$
A remark is warranted here. Things aren’t that simple in general. In many
countries, corporations are supposed to be taxed only as a form of withholding
before the individual pays his or her taxes. That is, the individual receives a
tax credit for the taxes that the corporation pays. Corporation taxation would
otherwise be considered a “double taxation.” Only “physical” persons are to be
taxed. In that case, Modigliani-Miller irrelevance will be restored, taxes do not
affect the value of the firm.

Of course, it is generally not true that cash flows are riskfree. We will therefore
consider a numerical example with risky debt, and study the effect of leverage on
the value of the firm. The example will illustrate that the possibility of default
does not overturn the optimality of 100% debt.

Example
We’ll revisit the example from the previous chapter and investigate what happens when the
JackMo receives a tax incentive of 50% of any interest payment. In addition to the other
data we are told that JackMo is paying 50% tax. We’ll first have to verify that the loan
payment (i.e., $D$) won’t change, to make sure we do not have a confounding factor. We
then conclude that both $V$ and $E$ increase.

Let us first look at the calculation of tax for Jackmo

<table>
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<tr>
<th>State</th>
<th>Taxable income</th>
<th>Tax</th>
<th>Pretax Cashflow</th>
<th>Aftertax Cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1000(0.80 − 0.55 − 0.1) − 5 = 145</td>
<td>72.5</td>
<td>700</td>
<td>627.5</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>1000(0.50 − 0.55 − 0.1) − 5 = −155</td>
<td>0</td>
<td>400</td>
<td>400</td>
</tr>
</tbody>
</table>

$D = 200$:

<table>
<thead>
<tr>
<th>State</th>
<th>Non-debt cashflow</th>
<th>Debt payment</th>
<th>Debt tax shield</th>
<th>Equity cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{u}$</td>
<td>627.5</td>
<td>220</td>
<td>10</td>
<td>417.5</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>400.0</td>
<td>220</td>
<td>0</td>
<td>180</td>
</tr>
</tbody>
</table>

$B^{200} = \frac{1}{1.1} (p^u 417.5 + (1 − p^u)180) = \frac{1}{1.1} ((0.167)417.5 + (1 − 0.167)180) \sim 200$

$D = 400$:

<table>
<thead>
<tr>
<th>State</th>
<th>Non-debt cashflow</th>
<th>Debt payment</th>
<th>Debt tax shield</th>
<th>Equity cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{u}$</td>
<td>627.5</td>
<td>640</td>
<td>120</td>
<td>107.5</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>400.0</td>
<td>400</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$B^{400} = \frac{p^u 107.5}{1.1} \sim 16$

Based on these cashflows we can find the values of the debt and equity.
18.3 Bankruptcy Costs

Default *per se* does not overturn the optimality of 100% debt, induced by the tax subsidy on debt financing. The previous example illustrates this. All it does is to change the *ownership* of the assets of the firm. *Bankruptcy costs*, however, put a drag on the *cash flow* of the firm available to traditional creditors in the event of default. These costs accrue to lawyers, accountants, the judicial system, various collection agencies, etc., all of whom are not traditional creditors. Since bankruptcy costs reduce the potential cash flow to traditional creditors, it lowers firm value. Trading off bankruptcy costs and debt tax shield will produce an optimal capital structure, as illustrated in figure 18.1.

**Example**

We will illustrate this with a variation on the previous JackMo example. Suppose now that JackMo does not sell his oil until two periods from now. The following is the evolution of possible (spot) prices for jet fuel:

\[
\begin{array}{ccc}
B & E & V \\
200 & 200 & 400 \\
400 & 16 & 416 \\
\end{array}
\]

Note that the higher debt is optimal.
The riskfree rate remains 10% throughout. The futures quote for delivery on the day after tomorrow is $0.4838. JackMo pays tax with 50%. In the case of bankruptcy there are payable costs of 5. We assume that the state-price probabilities remain the same, whether you end in the high state (jet fuel spot price of 0.80/Litre) or the low one.

Let us now suppose that First Yellow Bank makes loan offers based on the following total obligations of principal and interest:

1. $D = 200$, 
2. $D = 300$, and
3. $D = 310$.

and use the results to find which of these three debt levels maximizes the value of the firm, $V$.

Let us first summarize the cash flows for the firm before considering any loans.

<table>
<thead>
<tr>
<th>State</th>
<th>Investment cost/l</th>
<th>Fuel price/l</th>
<th>Delivery cost/l</th>
<th>Units Sold</th>
<th>Taxable income</th>
<th>Tax</th>
<th>Aftertax cashflow (time 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$uu$</td>
<td>5</td>
<td>0.4</td>
<td>0.9</td>
<td>0.1</td>
<td>1000</td>
<td>395</td>
<td>198</td>
</tr>
<tr>
<td>$ud$</td>
<td>5</td>
<td>0.4</td>
<td>0.6</td>
<td>0.1</td>
<td>1000</td>
<td>95</td>
<td>48</td>
</tr>
<tr>
<td>$dd$</td>
<td>5</td>
<td>0.4</td>
<td>0.4</td>
<td>0.1</td>
<td>1000</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that loan payment obligations of both 200 and 300 are risk free, because the company has enough cashflows at time 2. For these two cases we can calculate the loan amounts as

$B^{200} = \frac{200}{1.1^2} = 165.28$

$B^{300} = \frac{300}{1.1^2} = 247.93$

The problem is finding the loan amount with a $310 total obligation of principal and interest. Note that when the cash flow of the firm is only 300 (the $dd$ state), the firm is bankrupt,
and the available cash flow will be lowered with 5, the bankruptcy cost. The bank loan is, therefore, a security with the following possible cashflows at time 2:

<table>
<thead>
<tr>
<th>State</th>
<th>Cashflow at time 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>uu</td>
<td>310</td>
</tr>
<tr>
<td>ud</td>
<td>310</td>
</tr>
<tr>
<td>dd</td>
<td>295</td>
</tr>
</tbody>
</table>

The value of such a security can be found using the state price probabilities framework. We first have to find state price probabilities, where the states are defined by the fuel price. We will use the futures price at time 2 to do this. Investing in a futures has NPV equal to zero (see chapter 21) and we can therefore calculate the state price from

\[
0 = \frac{1}{1 + r} \left( p^u \left( \frac{1}{1 + r} (p^u X^{uu} + (1 - p^u) X^{ud}) \right) 
+ \left( (1 - p^u) \left( \frac{1}{1 + r} (p^u X^{ud} + (1 - p^u) X^{dd}) \right) \right) \right)
\]

where \( X^s \) is the cashflow from the futures investment in state \( s \), and we have used the assumption that the state price probability \( p^u \) remains constant through time. Solving this for \( p^u \), one obtains

\[
p^u = 0.66
\]

The loan with debt and interest obligation \( (D) \) equal to 310 is thus found as

\[
B_{310} = \left( \frac{1}{1 + r} \right)^2 (p^u 310 + (1 - p^u) (p^u 310 + (1 - p^u) 295)) = 251
\]

Note that this means the firm is effectively paying a 11.1% interest rate (compounded over two periods) on its loan.

Let us now calculate the time 2 cash flows to debt and equity for the three different loans.

\( D = 200: \)

<table>
<thead>
<tr>
<th>State</th>
<th>Aftertax cashflow</th>
<th>Debt payment</th>
<th>Tax shield</th>
<th>Equity cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>uu</td>
<td>602</td>
<td>200</td>
<td>17.5</td>
<td>419.5</td>
</tr>
<tr>
<td>ud</td>
<td>453</td>
<td>200</td>
<td>17.5</td>
<td>270.5</td>
</tr>
<tr>
<td>dd</td>
<td>300</td>
<td>200</td>
<td>17.5</td>
<td>117.5</td>
</tr>
</tbody>
</table>

\( D = 300: \)

<table>
<thead>
<tr>
<th>State</th>
<th>Aftertax cashflow</th>
<th>Debt payment</th>
<th>Tax shield</th>
<th>Equity Cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>uu</td>
<td>602</td>
<td>300</td>
<td>26</td>
<td>329</td>
</tr>
<tr>
<td>ud</td>
<td>453</td>
<td>300</td>
<td>26</td>
<td>179</td>
</tr>
<tr>
<td>dd</td>
<td>300</td>
<td>300</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\( D = 310: \)
18.4 Agency Costs

Besides bankruptcy costs, there are agency costs. In particular, managers may want to do things that are not in the interest of the traditional creditors. The ensuing drag on cash flow may be avoided in part by issuing debt, which forces management to go to the capital markets regularly to refinance debt.

18.5 Personal Taxes

Let's look at an extreme case, where interest payments on debt is taxed at the personal level, at a rate \( \tau_p \), whereas income from equity can be shielded from taxation at the personal level. In other words, the after-tax income from the coupon equals \( C(1 - \tau_p) \).

In the Modigliani-Miller model earlier in this chapter, we obtain:

\[
V_L = V_U + B_L \frac{\tau_e - \tau_p}{1 - \tau_p}.
\]  

(18.1)

If income from equity is also taxed at the personal level, at a rate \( \tau_e \), then

\[
V_L = V_U + B_L \left( 1 - \frac{(1 - \tau_e)(1 - \tau_p)}{1 - \tau_p} \right).
\]

(18.2)

18.6 General Equilibrium Effects Restore Irrelevance

"General equilibrium" effects are the consequences at the market level of systematic actions at the individual (company, investor) level. Miller (1977) studied
the effect of the issuance of corporate bonds when personal income taxation is progressive, i.e., \( \tau_p \) increases with income. Firms have an incentive to issue debt, until there are no more individuals in the economy with \( \tau_p < \tau_c \). At that point, there is no more advantage to issuing debt, because the marginal investor pays \( \tau_p = \tau_c \), so that the corporate debt tax shield is offset by the personal tax on coupon income. Then, at the margin, the individual company will be indifferent between issuing debt and equity.

Capital structure irrelevance restored!

References

Modigliani and Miller (1963) is the original reference to the tax effects. Miller (1977) studies the equilibrium arguments. An academic survey of these issues are in Swoboda and Zechner (1995). See Brealey and Myers (2002) or Ross, Westerfield, and Jaffe (2005) for the typical textbook discussions.
Problems

18.1 Leverage [6]
A firm has expected net operating income ($X$) of $600. Its value as an unlevered firm ($V_U$) is $2,000. The firm is facing a tax rate of 40%. Suppose the firm changes its ratio of debt to equity ratio to equal 1. The cost of debt capital in this situation is 10%. Use the MM propositions to:

1. Calculate the after-tax cost of equity capital for both the levered and the unlevered firm.
2. Calculate the after-tax weighed average cost of capital for each.
3. Why is the cost of equity capital higher for the levered firm, but the weighted average cost of capital lower?

18.2 GTC [5]
Note: In the question you are asked to assume risk neutrality. This means that the state price probabilities are not colored by risk aversion (fear) so they are equal to the estimated probabilities in the question.
Good Time Co. is a regional chain department store. It will remain in business for one more year. The estimated probability of boom year is 60% and that of recession is 40%. It is projected that Good Time will have total cash flows of $250 million in a boom year and $100 million in a recession. Its required debt payment is $150 million per annum. Assume a one-period model. Assume risk neutrality and an annual discount rate of 12% for both the stock and the bond.

1. What is the total stock value of the firm?
2. If the total value of bond outstanding for Good Time is $108.93 million, what is the expected bankruptcy cost in the case of recession?
3. What is the total value of the firm?
4. What is the promised return on the bond?

18.3 Bond Issue [4]
An firm that is currently all-equity is subject to a 30% corporate tax rate. The firm’s equityholders require a 20% return. The firm’s initial market value is $3,500,000, and it has 175,000 shares outstanding. Suppose the firm issues $1 million of bonds at 10% and uses the proceeds to repurchase common stock. Assume there is no change in the cost of financial distress for the firm. According to MM, what is the new market value of the equity of the firm?
18.4 *LMN* [3]
LMN is currently all equity financed. The equity of the firm is worth 7 million. LMN is planning to issue bonds with a value of 4 million and a 10% coupon. LMN is paying 30% corporate tax. Individual investors are paying 20% tax on capital gains and dividends, and 25% tax on interest income. In a Miller equilibrium, what is the new value of the firm after the bond issue?

18.5 *Bond* [3]
A company is issuing a 3 year bond with a face value of 25 million and a coupon of 10%. The company is paying taxes with 28%. The company’s cost of capital is 15%. What is the value of the bond issue for the company?

18.6 *Tax Shield Value*
The general expression for the value of a leveraged firm in a world in which $\tau_S = 0$ is

$$VL = V_U + \left( \frac{1 - (1 - \tau_C)}{1 - \tau_B} \right) B - C(B)$$

where $V_U$ is the value of an unlevered firm, $\tau_C$ is the effective corporate tax rate for the firm, $\tau_B$ is the personal tax rate of the marginal bondholder, $B$ is the debt level of the firm, and $C(B)$ is the present value of the costs of financial distress for the firm as a function of its debt level. (Note: $C(B)$ encompasses all non-tax-related effects of leverage on the firm’s value.)

Assume all investors are risk neutral.

1. In their no-tax model, what do Modigliani and Miller assume about $\tau_C$, $\tau_B$ and $C(B)$? What do these assumptions imply about a firm’s optimal debt–equity ratio?

2. In their model that includes corporate taxes, what do Modigliani and Miller assume about $\tau_C$, $\tau_B$ and $C(B)$? What do these assumptions imply about a firm’s optimal debt–equity ratio?

3. Assume that IBM is certain to be able to use its interest deductions to reduce its corporate tax bill. What would the change in the value of IBM be if the company issued $1$ billion in debt and used the proceeds to repurchase equity? Assume that the personal tax rate on bond income is 20%, the corporate tax rate is 34%, and the costs of financial distress are zero.

18.7 *Infty.com* [4]
Infty.com will generate forever a before-tax cash flow of $15. The corporate tax rate is 50%. The risk free rate is 10%.
1. Value the firm if it is all-equity

2. Value the firm if it issues a perpetual bond with coupon $5.
Chapter 19

Valuation of Projects Financed Partly with Debt

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  19.2.1 Adjusted Present Value (APV) .................................. 185
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  19.2.3 Weighted Average Cost of Capital .......................... 186
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19.1 Adjusting for Taxes

As we saw in chapter 18, taxes induces the firm to take on debt. How can one take account of this tax subsidy in valuation calculations?

19.2 Three Strategies that have been Suggested

19.2.1 Adjusted Present Value (APV)

Imagine you live in Modigliani and Miller's world (described in Chapters 17 and 18), and there are no personal taxes, but only corporate taxes $\tau_c$. Then the value of a project (firm) that is partly financed by (perpetual) debt with principal $D$ can be obtained by adjusting the value of the unlevered project:

$$V_L = V_U + \tau_c B_L.$$
Hence, the value of the levered project can be obtained as the Adjusted Present Value (APV) of the unlevered project. The Modigliani-Miller adjustment term, $r_c B_L$, is highly specific to their world of perpetual, riskfree cash flows. But the general principle should be clear. When financing a project partly with debt one should adjust the value with the present value of any debt tax shield.

**Example**

A all-equity financed project costs 100 today and lasts for one period with aftertax cashflow of 121 next period. The NPV of the project is 10.

What if the firm issues (one period) debt of 50 to finance the project? The debt pays interest of 5%. The firm is facing a 29% tax rate. The NPV of project will now increase by the debt tax shield.

$$NPV = 10 + \frac{0.29 \cdot 0.05 \cdot 50}{1 + 0.05} = 10.69$$

### 19.2.2 Flow to Equity

The Flow-To-Equity (FTE) is defined to be the NPV of the cash flow to the equityholders. This doesn't seem any different from what we have been doing throughout this text, where we have emphasized the importance of computing the value of equity separately.

But proponents of FTE usually advocate a particular formula, namely,

$$FTE = \frac{\text{periodic cash flow to equityholders}}{r_E},$$

where $r_E$ is the "required" rate of return on equity which we derived in Chapter 18.

$$r_E = \frac{E_L + B_L(1 - r_c)}{E_L} r_{VU} - \frac{B_L}{E_L} r_B (1 - r_c).$$

Of course, this formula is valid only in Modigliani and Miller's world of perpetual, riskfree cash flows, no personal taxes and perpetual debt.

In Modigliani and Miller's world, where the assets are "in place," maximizing the value of equity leads to the same decisions as maximizing the value of the firm. Hence, APV and FTE will generate the same decisions.

### 19.2.3 Weighted Average Cost of Capital

The idea of Weighted Average Cost Of Capital (WACC) is simple. One should discount the cash flows of the unlevered project (firm) using a weighted average cost of capital that reflects the advantages of the debt tax shield. In fact, the Modigliani and Miller analysis is used to derive the WACC, $r_{wacc}$. 
If there are only corporate taxes, the simplest way to derive $r_{wacc}$ from the analysis in Chapter 18 is as follows. You want to obtain $V_L$ as the cash flows from the unlevered firm discounted at $r_{wacc}$. But that requirement can be re-written:

$$V_L = \frac{F(1 - \tau_c)}{r_{wacc}} = V_U \frac{r_{V_U}}{r_{wacc}}.$$  

Solve for $r_{wacc}$:

$$r_{wacc} = \frac{V_U}{V_L} r_{V_U}.$$  

Then solve for $r_{V_U}$ in:

$$r_E = \frac{E_L + B_L (1 - \tau_c)}{E_L} r_{V_U} - \frac{B_L}{E_L} r_B (1 - \tau_c).$$

Note that $E_L + B_L (1 - \tau_c) = V_U$, so:

$$r_{wacc} = \frac{V_U}{V_L} \left( \frac{E_L}{V_U} r_E + \frac{B_L}{V_U} r_B (1 - \tau_c) \right).$$

Therefore,

$$r_{wacc} = \frac{E_L}{E_L + B_L} r_E + \frac{B_L}{E_L + B_L} r_B (1 - \tau_c).$$  \hspace{1cm} (19.1)

### 19.3 The General Principle: Net Present Value Again

All the rules in the previous section are derived from the same general principle: Net Present Value (NPV). Remember what NPV is about: it discounts cash flows according to their time stamp and risk. If cash flows from the project are risky, one need to use the discount rate that adjusts for the risk appropriately. If the debt tax shield is riskfree (e.g., because accounting earnings are always higher than interest payment on debt), then the cash inflows it generates should be discounted at the riskfree rate. Very often, the debt tax shield introduces an option-like payoff. In that case, one should use the general valuation principle in (10.1). We will see an example of this when we look at the Chippawhip company in chapter 23.
Problems

19.1 Project [3]
A company is considering a project with the following after-tax cashflows:

<table>
<thead>
<tr>
<th>t</th>
<th>( X_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-150,000</td>
</tr>
<tr>
<td>1</td>
<td>60,000</td>
</tr>
<tr>
<td>2</td>
<td>60,000</td>
</tr>
<tr>
<td>3</td>
<td>60,000</td>
</tr>
</tbody>
</table>

If the project is all-equity financed it has a required rate of return of 15%. To finance the project the firm issues a 4 year bond with face value of 100,000 and an interest rate of 5%. Remaining investments are financed by the firm’s current operations. The company is facing a tax rate of 30%.
Determine the NPV of the project.

19.2 Bond Issue [2]
A company issues a one year, zero coupon bond with face value of 25 million. The bond has an interest rate of 8%. The company is paying tax with 28%. Determine the value of the debt tax shield.

19.3 NORSK [7]
NORSK, Inc., is valued at \( V = 100 \). Tomorrows value, \( V' \), will be either 150 (“up” state) or 50 (“down” state), with equal chance. NORSK is presently an all-equity firm, but they are considering issuing a corporate bond with face value $100 and coupon $10, because they were told that they can reduce their taxable earnings with the amount of the coupon. The corporate tax rate is 50%. According to their accountants, NORSK will have $10 in taxable earnings in the “up” state, and $5 in the “down” state. The risk free rate is 10%.

1. How does the value of the firm change upon the bond issue?
2. Value the bond. Use this to re-value the firm, now using the APV formula, unlike in your previous answer.
3. Why is there a discrepancy between the values of the (levered) firm you obtained in the previous two answers?
Chapter 20

And What About Dividends?

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20.1 Dividends

Dividends are cash payments made by corporations to their shareholders.¹ By dividend policy we understand the timing and amounts of dividend payments.

¹Some corporations will also issue non-cash dividends, so called stock dividends. We will not discuss these, we concentrate on cash dividends.
20.2 The Miller and Modigliani Argument

Just like the debt/equity case, Miller and Modigliani set the stage for all subsequent analysis of dividends. They showed a similar irrelevance result to their debt/equity result:

In the absence of taxation, and holding the investment policy of the firm fixed, the firm's dividend policy will not affect the value of the firm.

This is for the same reasons: Same cash flows - same value. The cashflows accruing to the company is decided by the investments of the firm, not the dividend policy.

As long as the investment policy of the firm is set, every dollar that is paid to shareholders but needed to keep the company going must be recovered in the form of debt or equity issues. Of course, the allocation of firm value across debtholders and shareholders will be affected if there are any capital structure changes as a consequence of dividend payments, but this does not affect the value of the firm.

20.3 Why Pay Dividends if only the IRS Gains?

Again, just like with debt/equity, taxation upsets the applecart for Miller and Modigliani. The nice dividend irrelevance result is only valid under assumptions of no taxes. In the US, and most other countries, corporate and personal taxation is very important for changing preferences for dividends. There are two main issues:

20.3.1 Double Taxation

In many countries, including the US, dividends are taxed “twice.” First there is the taxation of corporate earnings. The after tax results from these earnings are then reflected in the dividends and stock prices of the corporation. And then individuals are taxed on their dividends and returns from selling the corporation’s stocks. Double taxation!

There are various methods for offsetting this double taxation, but very few countries manage to avoid it altogether.

Example

One example where the tax system “almost” avoids double taxation is Norway. There, corporations are taxed on their earnings, but dividends are not taxed at all for personal investors. This is only “almost” because foreigners are taxed “at home” on their dividend
income from Norwegian stocks. The effect of this is of course that all foreigners tries to avoid holding Norwegian stocks around the ex dividend date.

20.3.2 Individual Preferences for Capital Gains

On the level of an individual investor there is a clear preference for capital gains relative to dividends. Capital gains are taxed only when realized, and only if you have not died yet. (Which should keep you from realizing gains until you die, thereby avoiding capital gains taxation altogether).

By that argument firms do a disservice to shareholders when paying dividends. If shareholders need the money, they can borrow against their shareholding, with the promise to pay back the loan when they die. Alternatively each individual investor may want to sell some small fraction of her shares to create "homemade dividends."

20.4 Does the Market Agree?

Does the market agree with this analysis? It's not clear whether the tax-paying American is the marginal investor. Stocks with different yields may also attract different clienteles.

20.4.1 The Ex–Day Drop

The clearest evidence for tax effects should come from the price drop on the ex-dividend day (the first day the stock trades without dividend). Elton and Gruber (1970) were the first to look at this. The argument is as follows: Individuals who are about to sell their shares can choose whether to sell the day before the ex dividend date, or the day after. In equilibrium stocks must be priced so that the marginal investor is indifferent between the two alternatives. Equation (20.1) equates the proceeds from the two alternative strategies.

\[ PB - \tau_G(P_B - P_0) = E[P_A] - \tau_G(E[P_A] - P_0) + D(1 - \tau_D) \]  

(20.1)

Here \( P_B \) is the stock price just before the ex day, \( P_A \) is the stock price just after the ex day, \( D \) is the dividend amount, \( \tau_G \) is the capital tax rate and \( \tau_D \) is the tax rate on dividends. Note that this will not hold exactly, since there is always some uncertainty about the price after, but on average it should be a reasonable approximation. Simplifying equation (20.1) gives

\[ \frac{P_B - E[P_A]}{D} = \frac{1 - \tau_D}{1 - \tau_G} \]
In their paper Elton and Gruber found the ex-day price drop to be 77.8% of the dividend on average. This was for the 4/1966 till 3/1967 period. Evidence from preferred stock, which have a higher dividend yield, reported in Eades et al. (1984), is similar. That sounds pretty convincing. But there is quite a bit of variation in the ex-day behavior, and Elton and Gruber did find the ex-price drop of high dividend yield stocks to be as high as 118% of the dividend on average.

20.4.2 Looking Beyond the Border for More Evidence

In Hong Kong neither dividends nor capital gains are taxed. If our story about taxes is correct, there ought not to be any difference between the ex-day price drop and the dividend. But the numbers, taken from Frank and Jagannathan (1998), show there to be only an ex-day price drop of 50% of the dividend. Frank and Jagannathan claim transaction costs and bid-ask spreads alone explain the effect. Given this Hong Kong evidence, the US evidence do not necessarily have to be interpreted in terms of tax effects either.

20.4.3 The Evidence from Returns on Investment

In the seventies several academics compared the average return on high (dividend) yield stock portfolios to that of low (dividend) yield portfolios. Notable examples are Black and Scholes (1974) and Litzenberger and Ramaswamy (1979). Tax effects should manifest itself in higher average returns on high-yield portfolios. (Can you see why?) There is ample evidence that there is no discernible tax effects in these data.

20.5 Another Technique to Get Rid of Excess Cash: Share Repurchases

If a firm really has too much cash and believes that dividend taxation is a problem, it will do better by repurchasing stock. The stockholders would only be taxed at the (usually lower) capital gains rate, and have the option not to participate in the share repurchase, avoiding taxation altogether. Since share repurchases empirically lead to significant share price increases, the option not to participate is valuable.

In fact, recent evidence suggests that US corporations have “woken up” to this fact, the amounts of cash transferred from corporations to shareholders through
corporate repurchases have increased significantly, while the amounts distributed as dividends have stayed constant, or even gone down in real terms.\footnote{See Grullon and Michaely (2000).}

20.6 So, Why do Firms Pay Dividends? The Signalling Hypothesis

If it is costly to pay dividends, why do firms continue to do so? So far our only explanation of dividend behaviour has involved taxes. But there is another important assumption underlying the Miller Modigliani argument that may be violated. This is the \textit{symmetric information} assumption, that the market has the same information the management of the firm have, and therefore the same perception of firm value. But in practice the firm's management may be somewhat better than the market in estimating the true value of the firm. This leads to the following smart argument, for which there is solid evidence:

\begin{quote}
Since double taxation makes it is costly to pay dividends, only the richer (better) firms can afford to do so. Poorer firms can not. Hence dividend payments represent a unique opportunity to \textit{signal} to the market that your firm is better off than the average firm.
\end{quote}

There is not only casual evidence in favour of this signalling hypothesis (e.g. Apple Computer stopped paying dividends when they got into trouble.). The hypothesis is also supported by empirical studies showing that a firm's share price increases significantly upon announcement of dividend increases, despite the cost. Note that the signalling explanation can not be used \textit{independent} of the tax explanation. This because it relies on the presence of a costly signal. Dividend taxation plays the role of signal cost.

Note that the signalling hypothesis has similar implication for returns as taxes. Why would there be possible confusion between tax and signaling effects? The really attentive student will notice that any confusion would violate implication 2 of the Efficient Markets Hypothesis in Chapter 4.

20.7 Irrelevance Again

In fact, Miller and Scholes (1978) argued that it is possible to avoid dividend taxation entirely. It's ingenious, but works only in perfect capital markets. The details would take to long to go into, but the basic idea is offset dividend payments with interest on borrowed capital. The borrowed capital is to be invested in taxfree securities.
References

The original paper on dividend policy is Miller and Modigliani (1961). For the signalling hypothesis of dividend policy, see Bhattacharya (1979) and Miller and Rock (1985). For some work on the ex-day effect see Boyd and Jagannathan (1993). An academic survey of research on the dividend issue is found in Allen and Michaely (1995).
Problems

20.1 Dividend Inc [2]
The stock price of Dividend, Inc is 30 November 139. 1 December the stock goes ex dividend, paying a cash dividend of 11. The owners of stock in Dividend, Inc do not pay tax on their dividend income. What is your estimate of the stock price 2 December?

20.2 Dividends and Taxes [4]
A given share is sold for $30 just before time $t_0$. If the firm pays a $3 dividend per share, the price will immediately drop to $27. Suppose you own 100 shares. If the firm decides not to distribute dividends, you would need to sell 10 shares (at $30 a share) since you need to have a $300 cash income (pre tax) Assume that the shares were originally bought for $20 each.

1. If both ordinary personal tax rate and capital gains tax are 28%, what is your after-tax wealth under the two alternative situations?

2. Suppose now the tax rates for capital gains are lower, you pay 40% tax on ordinary income, and 16% tax on capital gains. What is your after-tax wealth under the two alternative situations?

20.3 Dividends [4]
The firm is capitalized by 100,000 shares of common stock which trade at the beginning of the period at 10 per share. The expected net income in period one is $X_1 = 200,000$ and the firm has declared a cash dividend of 1 per share, to be paid at the end of the period. The firm's cost of equity is 20%. Ignore personal taxes.

1. What is the ex-dividend share price? What would have been the end-of-period stock price if the firm skipped the dividend?

2. How many shares of commons stock will the firm have to sell at the ex-dividend price in order to undertake an investment project which requires an investment $I_1 = 200,000$?

3. What is the value of the firm just after the new issue? What would have been the value of the firm if it skipped the dividend and used the retained earnings to finance the investment?

20.4 Stable Rest [3]
Stable, Inc has for the last ten years been paying out 5% of the book value of equity as dividend. Rest, Inc has for the last fifteen years been paying 80% of after tax
income as dividend. This year both Stable, Inc and Rest, Inc is lowering dividend payments by $10.

If the signalling hypothesis is correct, which of the following statements is correct?

1. The stock price of Stable, Inc goes down by less than the dividend amount.
2. The market thinks profitability of Stable, Inc is lessened.
3. The market thinks profitability of Rest, Inc is improved.

20.5 Dividend Amount [4]
A company is expecting after tax income of 3 million next year. The company currently has a debt/equity ratio of 80%. The company has the possibility of investing in a project with at total investment of 4 million. If the company wants to keep its current debt/equity ratio, how much should the company pay as dividend next year?

20.6 Stock [2]
The current price of a stock is 50. The cost of capital for the company is 10%, and the company dividend is expected to grow by 5% annually. What is the expected dividend payment next period?
Part V

Risk Management
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Chapter 21

Risk and Incentive Management

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21.1 Hedging

In this chapter we look at corporate hedging. We will define hedging to be any action that reduces the variability of a corporation's cash flows. The opposite side of the coin is speculation, actions that increase the variability of cashflows. Most hedging decisions are close to zero NPV investments.
21.2 Hedging with Readily Available Contracts

21.2.1 Forward and Futures Contracts

Two prime examples of instruments for hedging are forward contracts and futures contracts. Both of these are agreements to buy (sell) given amounts of an underlying security at given prices (forward prices) and at given times (expiry dates). The forward price is set so that no money changes hand until the date of delivery. Hence, as already mentioned on a number of previous occasions, the initial value of the forward contract equals zero. The difference between forwards and futures is mainly that forwards are contracts between two counterparties, whereas futures are traded on organized exchanges. In addition to the place of trade, a futures contract has the distinguishing feature that the profit and losses from the contracts are settled during the life of the contract. This is called marking to market of a futures contract. For our purposes this difference is not particularly important, and we will mainly work with forward prices. In practice the fact that a security is traded on an exchange, with its implications for liquidity, price observability, and lack of counterparty risk is important.

Somebody who takes the delivery side of a futures or forward contract (promises to deliver the good in the contract) is said to go short. Somebody who takes the cash side is said to go long.

21.2.2 Pricing of Forward and Futures Contracts

How do we find the current forward price? For simplicity we consider a forward contract on an underlying asset that provides no income. There are also no restrictions on shortselling of the underlying asset. Let $S_t$ and $S_T$ denote the price of the underlying asset at $t$ and $T$, respectively. $r$ denotes the riskfree rate. Then the (time-$t$) forward price $F_t$ for a contract with deliver date $T$ has to satisfy

$$F_t = S_t(1 + r)^{(T-t)}$$

i.e., the forward price is the future value of the current price of the underlying. It is easy to show that violations of this will lead to free lunches. Let us start with the case where

$$F_t > S_t(1 + r)^{(T-t)}$$

Table 21.1 illustrates how we would set up a portfolio to exploit this free lunch.

On the other hand, if $F_t < S_t(1 + r)^{(T-t)}$, it is also easy to exploit the free lunch, as table 21.2 illustrates.
21.2 Hedging with Readily Available Contracts

Table 21.1 Arbitrage strategy for case $F_t > S_t(1 + r)^{(T-t)}$

<table>
<thead>
<tr>
<th>Time</th>
<th>Sell forward</th>
<th>Borrow $S_t$</th>
<th>Buy underlying</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>0</td>
<td>$F_t - S_t$</td>
<td>$-S_t(1 + r)^{(T-t)}$</td>
<td>$F_t - S_t(1 + r)^{(T-t)} &gt; 0$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F_t$</td>
<td>$-S_t$</td>
<td>$S_T$</td>
<td></td>
</tr>
</tbody>
</table>

Table 21.2 Arbitrage strategy for case $F_t < S_t(1 + r)^{(T-t)}$

<table>
<thead>
<tr>
<th>Time</th>
<th>Buy forward</th>
<th>Invest $S$</th>
<th>Short underlying</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>0</td>
<td>$S_T - F_t$</td>
<td>$-S_t(1 + r)^{(T-t)}$</td>
<td>$S_t(1 + r)^{(T-t)} - F_t &gt; 0$</td>
</tr>
<tr>
<td>$T$</td>
<td>$S_T$</td>
<td>$-S_T$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To avoid arbitrage we need an exact inequality

$$F_t = S_t(1 + r)^{(T-t)}$$

In actual markets there is a number of imperfections that will prevent this inequality from holding exactly. If it for example is not possible to short the underlying asset, one of the arbitrage arguments above do not work (can you see which?) Similarly, if there are costs of storing a physical underlying asset this will also enter into the pricing relationship.

An alternative formulation will show the same result: Futures and forward contracts are like bets. They do not require any investment when the contracts are “bought” (positions are entered into). In fact, the futures or forward price changes to the point that nobody has an incentive anymore to take further positions. Implication: The futures or forward price is set such that the value of the futures or forward contract is zero.

The alternative formulation works as follows. Use the result that the price (value) of a security equals the discounted expectation of its payoff (under the state price probabilities). In this case, the price equals zero. So:

$$0 = \frac{1}{(1 + r)^{T-t}} E^*[S_T - F_t].$$
Solving for the forward price, we obtain:

\[ F_t = E^*[S_T]. \]

Now apply the pricing formula to the underlying security:

\[ S_t = \frac{1}{(1 + r)^{T-t}} E^*[S_T]. \]

Plug this into the formula for the forward price, to obtain:

\[ F_t = S_t(1 + r)^{T-t}, \]

as before.

### 21.2.3 Options

We have already spent a lot of time on option contracts. Options are of course well suited for hedging, typically to insure against negative outcomes while keeping an upside potential.

### 21.2.4 Other Derivatives

Financial markets have developed a plethora of alternative derivatives to manage specific risk. Examples include swaps, Asian options, Bermudan options, Russian options, …

### 21.3 Synthetic Static Hedges

If no forward contract is available, it's possible to generate an artificial one. The artificial forward can be constructed at a single point in time and need not be adjusted later on. We will discuss an example showing how one can create an artificial forward foreign exchange (FX) contract.

#### Example

A US corporation has just entered a contract which promises DEM 100 three months from now (DEM is an imaginary currency). How can the corporation guarantee now the amount in USD they receive in 3 months, without using a currency forward contract? The current spot exchange rate and interest rates are

| DEM/USD spot rate | 4.00 |
| DEM 3 months interest rate | 5% |
| USD 3 months interest rate | 6% |

The following table illustrates the sequence of transactions necessary
### 21.4 Synthetic Dynamic Hedges

Risk that is nonlinear in the payoff on traded securities cannot be covered with a static hedge. The hedge portfolio has to be adjusted continuously. The frequency of rebalancing is dictated by the length of the time interval over which the relation between the risk and the underlying security becomes linear. Black and Scholes (1973) were the pioneers of this type of hedging. They exploited it to price options using a pure arbitrage argument.

Dynamic hedging is like flying a B-1 bomber. Using wing flaps, the airplane is continuously rebalanced. If one of the flaps gets stuck, the airplane is likely to crash. Analogously, the positions in a hedge portfolio have to be adjusted continuously. If one of the positions gets stuck, as we will see later in the Metallgesellschaft case, the hedge is likely to blow up. The analogy goes further: the re-balancing can be computed automatically. The risks of a bad hedge are enormous, just like the risk of flying a fly-by-wire Airbus with faulty navigational programs.

#### Example

To illustrate how one has to continually adjust hedges when hedging nonlinear payoffs, let us look at a simple, two period binomial example. The underlying stock is currently at $S_0 = 200$. We also know that $u = 1.02$, $d = 0.99$ and $r = 1\%$, which gives the following evolution of the stock price:

<table>
<thead>
<tr>
<th>Time</th>
<th>DEM</th>
<th>USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Now</td>
<td>Borrow in DEM: $100/1.0125 = 98.76$</td>
<td>Convert to USD: $-98.76$</td>
</tr>
<tr>
<td></td>
<td>Net Cashflows: $0$</td>
<td>Net Cashflows: $0$</td>
</tr>
<tr>
<td>3 months</td>
<td>Receive: $100$</td>
<td>USD Investment: $-98.76 \cdot 1.0125 = -100$</td>
</tr>
<tr>
<td>from now</td>
<td>Loan Repayment: $-98.76 \cdot 1.0125 = -100$</td>
<td>Net Cashflows: $0$</td>
</tr>
</tbody>
</table>

We leave it to the reader to calculate the forward price on the USD/DEM exchange rate “implicit” in this sequence of transactions.\(^1\)

---

\(^1\) These kinds of arguments are used to show an arbitrage result in FX markets, the Interest Rate Parity Theorem.
We want to hedge a two period call option with exercise price $K = 200$. If we first find the current price of the option:

$$\begin{align*}
P^u &= \frac{1.01 - 0.99}{1.02 - 0.99} = 0.67 \\
C^u &= \frac{1}{1.01} (0.67 \cdot 8.08 + 0.33 \cdot 1.96) = 6.06 \\
C^d &= \frac{1}{1.01} (0.67 \cdot 1.96 + 0.33 \cdot 0) = 1.30 \\
C_0 &= \frac{1}{1.01} (0.67 \cdot 6.06 + 0.33 \cdot 1.30) = 4.44
\end{align*}$$

Recall how, when we found the general method for pricing an option in the binomial setting, we calculated the hedge portfolio, a portfolio of the underlying security and risk free borrowing and lending that duplicated the payoff of the option. If $w_S$ and $w_p$ are the
number of units of the underlying and zero coupon bonds we sell, respectively, they are found from solving the following system of equations (we sell because we want to hedge the call, which means to insure the payoff on the call):

\[
\begin{align*}
C^u &= w_s S^u + w_F (1 + r) \\
C^d &= w_s S^d + w_F (1 + r)
\end{align*}
\]

Solving for \(w_s\) and \(w_F\):

\[
w_s = \frac{C^u - C^d}{S^u - S^d}
\]

\[
w_F = \frac{1}{1 + r} (C^u - w_s S^u) = \frac{1}{1 + r} (C^d - w_s S^d)
\]

Let us now see how these hedge ratios evolve as the call value and the value of the underlying changes through time:

\[
w^u_s = 1
\]

\[
w^u_F = -198.0
\]

\[
w^d_s = 0.33
\]

\[
w^d_F = -64.7
\]

To hedge dynamically one needs to continually update the positions in the underlying and the risk free asset. Suppose one did not do that. What happens?

Let us start with a portfolio that is hedged against changes in the first period. We leave it to the reader to show that a portfolio consisting of one bought call option, short 0.79 stocks and a risk free investment of 154.2 has constant cashflows at time 1. What if we take no action at time 1? What are the three possible values of the portfolio at time 2?

\[
\begin{align*}
uu : & \quad 1 \cdot 8.08 - 0.79 \cdot 208.08 + 154.2(1.01) = 1.0 \\
u d : & \quad 1 \cdot 1.96 - 0.79 \cdot 201.96 + 154.2(1.01) = -0.3 \\
\quad dd : & \quad 1 \cdot 0 - 0.79 \cdot 196.02 + 154.2(1.01) = 2.4
\end{align*}
\]

The payments at time 2 obviously differ across the possible states. The reader should show how the hedge ratios ought to change at time 1, and how this can guarantee the contingent payoff at time 2.

The example should convince one that dynamic hedging is dangerous in the hands of the uninitiated.

21.5 The Metallgesellschaft Case

Let us look at a real-life dynamic hedge that “crashed”, the Metallgesellschaft case. At the start of 1994 Metallgesellschaft A.G, the 14th largest corporation in
Germany, publicly acknowledged potential losses of USD 1 billion from trading in oil futures. The losses were coming from a US subsidiary, MG Corp and its division MG Refining and Marketing (hereafter MGRM).

During 1992-93 MGRM had signed a large number of contracts for delivering oil on a long term basis, up to 10 years, at a fixed price. MGRM had thus committed itself to supplying a flow of the underlying in the future. MGRM had no competitive advantage in supplying oil, no oil in the ground. The MGRM plan was a marketing plan, the long term fixed price contracts were used as a marketing tool to get business. The long term obligations were to be hedged in financial markets. To hedge their future short positions in the underlying, MGRM entered into a strategy of rolling over long positions in short term futures contracts.

In 1993 the oil price started falling. At the same time the basis, the difference between futures prices and spot prices started changing. Both of these events induced large losses on the futures position. By December '93 the oil price was down to 14.41 from 20.16 in March of '93. MGRM were facing huge losses on their futures contracts, and needed huge inflows of cash to meet margin calls in the futures markets. Finally Metallgesellschaft closed down the whole operation, liquidated the derivatives contracts and settled their long term fixed price contracts. A number of people have pointed out that this closing of positions was the worst thing to do: the losses on futures contracts would have been (mostly) offset by the future gains from the fixed price contracts.

Let us first show how a (simplified version of the MGRM) rolling short term futures hedge can be used to offset a flow of the underlying. Suppose MGRM has an obligation to deliver 10 mill bbl of oil each year for the next five years. The contract price is $20 per barrel. The oil has to be bought in the spot market each year. This contract will (ex post) be unprofitable if the spot prices are above the fixed price of $20. MGRM will hedge this risk by buying annual futures contracts, and plan to "roll over" the short term contracts in a particular manner.

Their procedure for "rolling over" is to start with an outstanding futures amount equal to the sum of their future delivery obligations. In this case MGRM starts with long futures of 50 mill bbl, and decrease the quantity by the amount delivered each year.

The annual delivery and the futures outstanding is illustrated in the next figure.
To illustrate how this hedge work, make some further simplifying assumptions. All payments are supposed to be made at year-end. The spot price equals the futures price. We ignore marking to market and treat the contracts as forward contracts.

What if the spot price at the beginning of the hedge is 20, and it then goes down to 17 at the end of year 1 and stays there for the next 4 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell oil (fix price)</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>( (10 \cdot 20) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy oil (future)</td>
<td>-200</td>
<td>-170</td>
<td>-170</td>
<td>-170</td>
<td>-170</td>
</tr>
<tr>
<td>( (10 \cdot 20) )</td>
<td>( (10 \cdot 17) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>“Roll over” hedge</td>
<td>-120</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( (40(17 - 20)) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As the table shows, there is a large loss at time 1 from the futures contract, but this will be offset by future gains from the difference between the fixed price and the futures price. For this particular outcome the roll over strategy implies large demands for cash at time 1, to cover the losses from the futures. In a futures market these losses will be met with margin calls on the remaining futures.

As the (simplified) example shows, MGRM should not have closed its position at time 1. If all positions are closed at time 1, a large loss is realized, and there
is no potential for future gains. But the MG case was not that simple. In 1993 the relationship between futures and spot prices flipped between *backwardation* and *contango*. (In backwardation the spot price is higher than the futures price, contango is when the futures price is higher than the spot price.) This added further losses to the already substantial losses MG had. The basis change induced losses for the company that it did not recuperate.

The thing to note about the rolling over strategy is the danger of large intermediate cash outflows. These may be recuperated, but the liquidity need may be a strain. This problem could have been mitigated by entering into longer term forward/futures contracts with a matching maturity to the future oil outflows. Such a long term forward strategy would not have had any intermediate liquidity needs.

### 21.6 Value at Risk (VaR)

To measure the risk of a portfolio with complex positions in derivatives, banks nowadays (have to) compute a number which is called their *Value at Risk* (VaR). Inspired by practice in the casualty insurance business, this number is the *maximum loss that the portfolio would run with a certain probability*, say, 95%. VaR is thus determined by the tail of the density of portfolio losses.

**Example**

To show an example of a VaR calculation we will for originality look at a Norwegian mutual fund which has a portfolio consisting of Norwegian Stocks closely matching the composition of the Oslo Stock Exchange. The portfolio has a total value of NOK 100 million. A broad based OSE index is the OSE All Share index, the evolution of which from 1983 to 1999 is shown in figure 21.1.

The mutual fund wants to answer the question: What is the “most” they can lose in one week? To do so we base the question on the empirical probability distribution of changes in the index over a week. Figure 21.2 summarizes this information, both as a histogram that approximates the frequency distribution and the empirical cumulative distribution.

The cumulative figure gives us the basis for answering the following question: *With 5% confidence, what is the expected maximum amount that we can lose?* From the figure, the cumulative probability distribution puts the 5% level around −0.045, or a −4.5% return on a weekly basis. Since 4.5% of 100 million is 4.5 million, your *Value at Risk* is 4.5 million. Another way to state the implication of the VaR calculation: Under normal market conditions, the most the portfolio can fall in value is 4.5 million.

In calculating VaR there are number of features to note. The VaR is based on a historical estimated probability distribution. The choice of methods for estimating this empirical probability distribution will have a large effect on the VaR calculation, since it is a tail probability, which is very sensitive to deviations
from normality. One is also implicitly making strong assumptions about the stationarity of the functional relationships when using historical data to estimate VaR. Further elaboration should be delegated to more specialized texts.

21.7 Should Corporations Hedge?

Why would corporations hedge using the various contracts discussed? As mentioned, most hedging transactions tend to be close to zero-NPV transactions. Hence they will not change the value of the firm. How is it then that so many corporations spend so much resources on corporate hedging? Reducing the variability of a corporation's cashflows will actually run counter to the preferences of equityholders, since this reduces the option value of their stocks!

21.8 Using Risk Management Techniques to Manage Incentives

We leave you with the following brain teaser. There is right now a war of computer alphabets between Sun (creator of Java) and Microsoft (creator of, well, you know what). Sun is in a relatively weak position, fighting against a company that has a virtual monopoly on the alphabet of computer software. So, if you are a software engineer, you may wonder whether it is worth investing in Java. If Sun loses the
Figure 21.2 Empirical probability distribution for OSE All share index
battle of alphabets, your human capital (tied to Java) would be worthless. Yet Sun needs engineers like you. If enough good engineers work for Sun, they may win the battle.

What could Sun do to entice you to move to sunny Silicon Valley, as opposed to rainy Seattle (besides using the weather as an nonpecuniary argument)?

References


See Jorion (2001) for a textbook treatment of Value at Risk. An overview of the statistical problems when using VaR is in Duffie and Pan (1997).

Problems

21.1 Binomial Options [4]
Consider a case where the stock price $S$ follows a binomial process. Currently, the stock price is $S_0 = 100$. Each period, the stock price either moves down 10% or up 15%. The (one period) risk free interest rate is 2.5%.

Consider a one-period call option with exercise price $X = 100$.

1. If you own one stock, how many call options do you need to buy at time 0 for the cashflows in period 1 to be riskless?

2. If you own one call, how many units of the underlying do you need to buy at time 0 for the cashflows in period 1 to be riskless?

Consider now a two period American Call option, with exercise price $X = 100$.

3. If you own one stock, how many of the two period call options do you need to buy at time 0 for the cashflows in period 1 to be riskless?

21.2 Option [4]
The current stock price is 160. Next period the price will be one of 150 or 175. The current risk free interest rate is 6%. You buy 1 stock and issue $m$ call options on the stock with an exercise price of 155. What must be $m$ be for the portfolio to be risk free?

21.3 Futures Price [5]
Suppose that storing the physical asset has a cost of $c$ in the period from $t$ to $T$. Show that the futures price satisfies

$$F = S(1 + r)^{(T-t)} + c$$

21.4 Hedging [3]
The stock price can next period be either 100 or 200. The stock price today is 150. You have a put option that expires next period. The exercise price of the put option is 100. The price of a discount bond that matures next period is 0.8. You would like to eliminate the risk of holding the put option. What position do you need to take in the stock?

21.5 VaR [4]
You own a portfolio with a value today of 10 million. The standard deviation of the weekly return is 0.01. Assuming that weekly returns are normally distributed, estimate Value at Risk (VaR) on a weekly basis with a confidence level of 5%.
21.6 VaR [3]
Your current portfolio of equities has a market value of 100,000. Assume normally distributed returns.

1. Suppose the whole portfolio is invested in one stock. The stock has an annual expected return of 10% and an annual standard deviation of 25%.
   Estimate Value at Risk for your portfolio on a daily horizon and a confidence level of 1%.

2. Suppose instead that the portfolio is invested with equal weights in two stocks, each with the same expected return and standard deviation 25%. If the correlation between the two shares is positive, but less than one, will the VaR of the portfolio be smaller or larger than the previous VaR? What if the correlation is negative?

21.7 Are You Lucky [5]
The current value of equity in the Are You Lucky Gold Mine (AYLGM) is 50. One period from now the stock price will be one of 58.2 or 45.8. The risk free interest rate is 5% per period.

1. Calculate the price of a call option on one AYLGM stock with exercise price 50.

2. What is the forward price for delivery of one AYLGM stock one period from now?

3. If you want to create a portfolio of forward contracts and risk free borrowing and lending that has the same payoffs as the call option with exercise price of 50, what is your position in forwards and risk free borrowing and lending?

21.8 T Bill Future [3]
Today, a Treasury bill (which carries no coupon) is selling for 91.5% (of par value). A futures contract for the delivery of the Treasury bill tomorrow carries a futures price of 91.5% (of par value). You don’t know the actual one-period risk free rate, but you know it is positive. Is there a free lunch?
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Part VI

Summary of the Insights
Chapter 22

Fourteen Insights

We have covered quite a bit of material. As a result, there is often the danger of not seeing the wood for the trees. Therefore, we decided to put the main insights together in a succinct way. Here they are.

1. The theory of finance works in an abstract world “without friction.” Know the axioms to understand the implications. The real world may be different, but a lot can be learned from knowing where it differs.

2. Financial markets are supermarkets for cash flows with different risk and time patterns.

3. The Efficient Markets Hypothesis really tells us that there won’t be abnormal returns once you adjust for risk. If someone has been consistently making more money than you, it’s because s/he has been taking more risk.

4. Question the habit of converting interest rates for different maturities into prices. You intuitively understand much better what it means for one price to be higher than another; it’s really hard to find meaning in the statement that, e.g., the yield on one bond is higher than on another.

5. The only valid way to value companies, securities or projects is to compute net present value. That is, determine future cash flows, multiply them with the prices of pure discount bonds (i.e., discount them appropriately), and add everything together (as if it were a basket of fruit).

6. When there is uncertainty, this principle still applies. In that case, you multiply the cash flows in each state of the world with the price of the corresponding state security, and add everything up. Technically, you compute
expected cash flows where you use state price probabilities and discount at the risk-free rate.

7. Because prices are really discounted expectations (with respect to state price probabilities), you can apply anything you know about expectations (e.g., Jensen's inequality) to determine what arbitrage-free prices look like. As a result, you can often find restrictions on prices without even having to know the state-price probabilities.

8. Don't let anybody talk you into a valuation procedure whereby you take expected future cash flows (where expectations are evaluated using "physical" probabilities) and discount them at some "risk adjusted" discount rate. This is still popular in many circles, but it's rather difficult to match with asset pricing theory. Besides, it does not force you to think about all the contingencies; it only invites you to think about the "average."

9. Only cash flows matter for valuation. Accounting earnings are relevant only because they determine taxes (which are cash flows).

10. The CAPM states that the required expected return on a security is not determined by its own return variability, but only to the extent that it contributes to the total risk (variance) of the market as a whole, i.e., only in proportion to its "beta." Of course, that applies only if investors care merely about variance, and not, e.g., skewness or leptokurtosis.

11. Equity (shares of stock) in a levered company effectively gives the equity holders an option-like stake in the company. Bondholders effectively write a put option to the equity holders. Warrants and convertible bonds avoid bondholders from being exploited by equity holders, who have the incentive to increase the risk of the company, instead of maximizing the value of the firm.

12. When assets are in place, it is hard to change the value of the firm by changing the capital structure. This only works if it changes payments to third parties (lawyers, the government, pension holders, etc...). This is the Modigliani-Miller result.

13. Cash dividends are in general a bad idea because equity holders end up paying more taxes. Stock repurchases are better if you must pay dividends.

14. It is possible to generate complicated payoff structures and hence to hedge complicated risk by dynamic replication: strategies whereby the composition of the hedge portfolio is changed periodically. This is the main insight of the Black-Scholes model.
Part VII

Longer Examples
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Chapter 23

Longer Examples

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In this part we show some worked out examples. The reader should try to do the problems before looking at the solutions.

23.1 Determining the Maximum Bid on a Gold Mining License

A gold mining company is considering bidding for a license to exploit a gold mine containing 10,000 pounds of gold. It costs $100,000 to open the mine tomorrow. Once opened, it costs $300 per ounce (Oz) to extract and ship the gold. Tomorrow's gold price will be either $350/Oz or $250/Oz. The riskfree rate is 10% and today's gold futures price is $300.

What is the maximum price to bid on the mining license?

Solution

The maximum bid is such that the NPV of the project is zero. Note that the decision to open the mine is delayed until "tomorrow." This decision can therefore be made contingent on the gold price. The mine should only be opened if the gold price is 350, in which case the cash flows are

\[(350 - 300)10,000 - 100,000 = 400,000\]
The maximal bid is the present value of this contingent cashflow. To actually value it, need to find the state price probability. Find this from the futures price

\[
0 = \frac{1}{1 + r} (p^u(350 - 300) + (1 - p^u)(250 - 300))
\]

implies

\[
p^u = 0.5
\]

The present value of the cashflows from the mine is calculated as

\[
PV = \frac{1}{1 + r} (p^u400,000 + (1 - p^u)0) = 181,818
\]

This is the maximal bid.

### 23.2 Chippawhip

Chippawip is a startup company, to engage in the production of a new computer memory chip. There will be a $4,000 initial outlay for production facilities. This ensures a production of 1,000 units per period, for 2 periods. Because of the competitiveness of the chip market, the total production can always be sold, but at varying prices.

Here is what the founders of Chippawip think are the possible scenarios for per-unit chip prices:

There is no time-0 price – the chip is not marketed yet. There are two possible period-1 prices and three possible period-2 prices. For simplicity, we refer to times \( t = 0 \) (when investment outlay is done), \( t = 1 \) (end of first period), \( t = 2 \) (end of second period).

We are also given
• Per-unit variable production cost is $1

• The corporate tax rate (on earnings after depreciation and interest expense) is 50%.

• Depreciation is straight-line over 2 periods (i.e., $2,000 per period).

• The time-0 prices of call options to purchase one memory chip, as a function of maturity and strike price:

<table>
<thead>
<tr>
<th>Expiration Date</th>
<th>Strike Price</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$1.011</td>
</tr>
</tbody>
</table>

• The price of a one-period zero-coupon bond with face value $1 is $0.95 in both periods.

• Dividend policy: in both states at $t = 1$, an amount equal to $1,750 is retained in the company, and invested at the risk-free rate; the proceeds are used to pay bondholders and shareholders at $t = 2$. Assume that interest on this investment is not taxed.

• Default: at $t = 1$, the company defaults if and only if it cannot pay the coupon out of current cash flow.

• Tax losses can not be carried forward.

Questions

1. Compute the value of Chippawip in each period and state.

2. The founders of Chippawip are considering a debt issue, to finance part of the initial investment outlay. They are thinking about a $3,500 (face value) straight debt issue with a coupon of 7%. Compute the value of the debt tax shield at each point in time.

3. Compute the time-0 value of the debt issue.

4. At each time and in each state, determine the dividends that Chippawip pays to its future shareholders. Dividends are to be paid out of current cash flow. Compute the time-0 value of equity in Chippawip, using these dividend data.
5. Also compute the time $t = 0$ value of equity using value additivity, i.e., the proposition that the value of equity must equal that of the firm minus the value of the bond issue.

6. How much equity do the founders of Chippawip have to raise?

Solution

We will first find the state price probabilities, which can be found from the option prices.

$$r = \frac{1}{0.95} = 5.263\%$$

Use the one-period option with exercise price 4 to find $p^u$:

$$0.38 = \frac{1}{1 + r} \left( p^u C_u + (1 - p^u)C_d \right) = 0.95 (p^u (5 - 4) + (1 - p^u)0)$$

$$p^u = \frac{0.38}{0.95} = 0.4$$

Use the two-period option with exercise price 5 to find $p^{uu}$:

$$0.289 = \frac{1}{1 + r} \left( p^{uu} C^{uu} + (1 - p^{uu})C^{ud} \right)$$

$$= (0.95)^2 (0.4 (p^{uu} 2 + (1 - p^{uu})0) + 0.6 \cdot 0)$$

$$p^{uu} = 0.4$$

Use the two-period option with exercise price 3 to find $p^{du}$:

$$1.011 = \frac{1}{1 + r} \left( p^{uu} \frac{1}{1 + r} (p^{uu} C^{uu} + (1 - p^{uu})C^{ud}) + (1 - p^{uu}) \frac{1}{1 + r} (p^{du} C^{du} + (1 - p^{du})C^{dd}) \right)$$

$$= (0.95)^2 0.4 (0.4(7 - 3) + 0.6(4 - 3)) + 0.6 (p^{du} (4 - 3) + 0)$$

$$p^{du} = 0.4$$

The state price probability is 0.4 in all "up" states.

Let us first find the value of the company ignoring debt and dividends.

Period 0: Invest 4000.
Period 1:

<table>
<thead>
<tr>
<th>State</th>
<th>Sales</th>
<th>Variable cost</th>
<th>Before tax cashflow</th>
<th>Depreciation</th>
<th>Taxable Income</th>
<th>Tax</th>
<th>After tax cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>5000</td>
<td>1000</td>
<td>4000</td>
<td>2000</td>
<td>2000</td>
<td>1000</td>
<td>3000</td>
</tr>
<tr>
<td>d</td>
<td>3000</td>
<td>1000</td>
<td>2000</td>
<td>2000</td>
<td>0</td>
<td>0</td>
<td>2000</td>
</tr>
</tbody>
</table>

Period 2:

<table>
<thead>
<tr>
<th>State</th>
<th>Sales</th>
<th>Variable cost</th>
<th>Before tax cashflow</th>
<th>Depreciation</th>
<th>Taxable Income</th>
<th>Tax</th>
<th>After tax cashflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>uu</td>
<td>7000</td>
<td>1000</td>
<td>6000</td>
<td>2000</td>
<td>4000</td>
<td>2000</td>
<td>4000</td>
</tr>
<tr>
<td>ud</td>
<td>4000</td>
<td>1000</td>
<td>3000</td>
<td>2000</td>
<td>1000</td>
<td>500</td>
<td>2500</td>
</tr>
<tr>
<td>dd</td>
<td>1000</td>
<td>1000</td>
<td>0</td>
<td>2000</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The cash flows in the various periods can thus be summarized as:

\[
X^{uu} = 4000 \\
X^u = 3000 \\
X^d = 2000 \\
X^{dd} = 0 \\
X^0 = -4000 \\
X^{ud} = 2500
\]

Calculate value as:

\[
V^u = 3000 + \frac{1}{1 + r} (p^{uu}4000 + (1 - p^{uu})2500) \\
= 3000 + 2945 = 5945 \\
V^d = 2000 + \frac{1}{1 + r} (p^{du}2500 + (1 - p^{du})0) \\
= 2000 + 950 = 2950 \\
V = \frac{1}{1 + r} (p^uV^u + (1 - p^u)V^d) \\
= 0.95 (0.4 \cdot 5945 + 0.6 \cdot 2950) \\
= 3940.60
\]

As we see, the value of the firm is less than the current investment of 4000. On this basis, the investment should not be made. But this ignores any interest tax shields.
The debt issue with face value of 3500 has interest payments of 245 in periods 1 and 2. Note that the company always has enough cashflow at time $t = 1$ to cover the interest payment. The only possibility of default is then at time $t = 2$. With interest, the retained earnings from time $t = 1$ is $1750 \cdot (1 + r) = 1842.10$. The only default is at time $t = 2$ in the $dd$ state, where the firm only has retained earnings to use towards the bond payments. The bond payments can therefore be summarized as

Let us first find the value of the bond, which is how much the bond is issued for today.

$$B^u = 245 + \frac{3745}{1 + r} = 3802.75$$

$$B^d = 245 + \frac{1}{1 + r} (p^u 3745 + (1 - p^u)1842.10) = 2718.1$$

$$B = \frac{1}{1 + r} (p^u 3802.75 + (1 - p^u)2718.10) = 2994.36$$

Let us now find the value of the tax shield, which is summarized as
23.2 Chippawhip

\[
\text{Shield}^u = 122.5 + \frac{122.5}{1 + r} = 238.87
\]

\[
\text{Shield}^d = 0 + \frac{1}{1 + r} (p^d u 122.5 + (1 - p^d)0) = 46.55
\]

\[
\text{Shield} = \frac{1}{1 + r} (p^u 238.8 + (1 - p^u)46.55) = 117.27
\]

The debt tax shield increases the value of the firm by 187.10, we get

\[
V = 3940.60 + 117.27 = 4057.87
\]

What are the cashflows to equity? We will show this using several methods. First we calculate the cash flows to equity before the tax shield and then add the value of the tax shield

\[
4000 + 1842.10 - 3745 = 2097.1
\]

\[
3000 - 1750 - 245 = 1005
\]

\[
2500 + 1842.10 - 3745 = 597.10
\]

\[
2000 - 1750 - 245 = 5
\]

\[
0
\]
The value of equity (before tax shield) is calculated as

$$E^u = 1005 + \frac{1}{1 + r} (p^{uu}u 2097.1 + (1 - p^{uu})597.10)$$

$$E^d = 5 + \frac{1}{1 + r} (p^{du}597.10 + (1 - p^{du})0) = 231.9$$

$$E = \frac{1}{1 + r} (p^u 2142.25 + (1 - p^u)231.90) = 946.23$$

Add the value of the tax shield to get correct value of equity.

$$E = 946.23 + 117.27 = 1063.5$$

Alternatively we calculate the cash flows to equity after the tax shield each period

$$4000 + 1842.10 - 3745 + 122.5 = 2219.60$$

$$3000 - 1750 - 245 + 122.5 = 1127.5$$

$$2500 + 1842.10 - 3745 + 122.5 = 719.60$$

$$2000 - 1750 - 245 = 5$$

$$E^u = 1127.5 + \frac{1}{1 + r} (p^{uu}u 2219.6 + (1 - p^{uu})719.60)$$

$$= 1127.5 + 1253.62 = 2381.12$$

$$E^d = 5 + \frac{1}{1 + r} (p^{du}719.60 + (1 - p^{du})0)$$

$$= 5 + 273.44 = 278.44$$

$$E = \frac{1}{1 + r} (p^u 2381.12 + (1 - p^u)278.44) = 1063.5$$

Let us finally find the value of equity using value additivity:

$$E = V - B = 4057.87 - 2994.36 = 1063.5$$

As we see, these calculations of the equity value give the same answer, which they should.
To finance the investment, equityholders in Chippawhip have to raise

\[ 4000 - B = 4000 - 2994.36 = 1005.64 \]

for which they get assets worth 1063.5 today. Investing in Chippawhip has a NPV of 1063.5 - 1005.64 = 57.9 for the equity investors.
Part VIII

Appendix
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Appendix

A Notation and Formulas

In this appendix we collect a list of the notation we use in the book, and some of the formulas.

A.1 Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>β</td>
<td>Beta</td>
</tr>
<tr>
<td>σ</td>
<td>Standard deviation</td>
</tr>
<tr>
<td>ρ</td>
<td>Correlation</td>
</tr>
<tr>
<td>ρ_{i,t}</td>
<td>Risk premium</td>
</tr>
<tr>
<td>ω</td>
<td>Portfolio weight</td>
</tr>
<tr>
<td>B</td>
<td>Bond value</td>
</tr>
<tr>
<td>C</td>
<td>Value of Call option</td>
</tr>
<tr>
<td>cov</td>
<td>Covariance</td>
</tr>
<tr>
<td>d</td>
<td>Down state</td>
</tr>
<tr>
<td>D</td>
<td>Debt (face value)</td>
</tr>
<tr>
<td>e</td>
<td>2.71828182846...</td>
</tr>
<tr>
<td>E[.]</td>
<td>Expectation</td>
</tr>
<tr>
<td>E[.]</td>
<td>I_t</td>
</tr>
<tr>
<td>F</td>
<td>Forward/futures price</td>
</tr>
<tr>
<td>FV_t</td>
<td>Future value</td>
</tr>
<tr>
<td>g</td>
<td>Growth of cashflows per period</td>
</tr>
<tr>
<td>i</td>
<td>Inflation</td>
</tr>
<tr>
<td>I^*</td>
<td>Price of Index Option</td>
</tr>
<tr>
<td>IRR</td>
<td>Internal rate of return</td>
</tr>
<tr>
<td>K</td>
<td>Strike price</td>
</tr>
</tbody>
</table>

233
\( \ln() \)  
Natural log, defined by \( \ln(e) = 1 \)

NOK  
Norwegian Kroner (currency)

NPV  
Net present value

\( P \)  
Value of Put option

\( P_t \)  
Present value of receiving one dollar at time \( t \).

\( p^s \)  
State price probability

PV  
Present value

\( r \)  
Interest rate/Return

\( r_f \)  
Risk free interest rate

\( r_t \)  
Interest rate for discounting cash flows in period \( t \)

\( S_t \)  
Price of underlying at time \( t \)

SD  
Standard deviation

\( t \)  
Time

\( T \)  
Maturity date

\( u \)  
Up state

USD  
US dollars (currency)

var  
Variance

\( W \)  
Warrant value

\( X_t \)  
Cash flow at time \( t \).

### A.2 Formulas

**Calculating returns**

\[
\text{Return} = \frac{(Value_t - Value_{t-1}) + Payments_t}{Value_{t-1}} = \frac{Payments_t}{(Value_{t-1})} - 1
\]

**Compounding**

\[
FV_t = PV \left( 1 + \frac{r}{n} \right)^{nt}
\]

\[
FV_t = PV(e^{rt})
\]

\[
PV = FV_t \left( 1 + \frac{r}{n} \right)^{-nt}
\]

\[
PV = FV_t(e^{-rt})
\]

**Present values**

\[
P_t = \left( \frac{1}{1 + r_t} \right)^t
\]
\[ r_t = P_t^{-\frac{1}{T}} - 1 \]

\[ PV = \sum_{t=1}^{\infty} P_t X_t \]

\[ PV = \sum_{t=1}^{\infty} \frac{X_t}{(1 + r_t)^t} \]

\[ PV = \sum_{t=1}^{\infty} \frac{X}{(1 + r)^t} = \frac{X}{r} \]

\[ PV = \sum_{t=1}^{\infty} \frac{X(1 + g)^{t-1}}{(1 + r)^t} = \frac{X(1 + g)}{r - g} \]

\[ PV = \sum_{t=1}^{T} \frac{X}{(1 - r)^t} = X \left[ \frac{1}{r} - \frac{1}{r(1 + r)^T} \right] \]

\[ NPV = PV - X_0 \]

**IRR**

\[ NPV = 0 = \sum_{t=1}^{T} \frac{X_t}{(1 + IRR)^t} - X_0 \]

**Stock valuation**

\[ P_0 = \sum_{t=1}^{\infty} \frac{E[\text{Dividend}_t]}{(1 + r)^t} \]

\[ P_0 = \frac{E[\text{Dividend}_1]}{r - g} \]

**Statistics**

\[ \text{var}(X) = \sigma^2(X) = E \left[ (X - E[X])^2 \right] \]

\[ \sigma(X) = \sqrt{\text{var}(X)} \]

\[ \text{var}(aX) = a^2 \text{var}(X) \quad (a \text{ constant}) \]

\[ \text{cov}(X, Y) = E \left[ (X - E[X])(Y - E[Y]) \right] \]

\[ \text{cov}(a, X) = 0 \quad (a \text{ constant}) \]
\[ \text{cov}(aX, bY) = ab \text{cov}(X, Y) \quad (a, b \text{ constants}) \]
\[ \text{var}(X + Y) = \text{var}(X) + 2 \text{cov}(X, Y) + \text{var}(Y) \]
\[ \text{corr}(X, Y) = \rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma(X)\sigma(Y)} \]

**Portfolio theory**

\[ \tilde{\bar{r}}_p = \sum_{j=1}^{N} \omega_j \tilde{r}_j \]
\[ \omega_j = \frac{\text{Market value of asset } j}{\text{Market value of all assets}} \]
\[ E[\tilde{\bar{r}}_p] = E \left[ \sum_{j=1}^{N} \omega_j \tilde{r}_j \right] = \sum_{j=1}^{N} \omega_j E[\tilde{r}_j] \]
\[ \sigma_p^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} \omega_j \omega_i \sigma_{ij} \]

**CAPM**

\[ E[\tilde{r}_j] = r_f + (E[\tilde{r}_m] - r_f) \beta_j \]
\[ \beta_i = \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)} \]
\[ \beta_p = \sum_{j=1}^{N} \omega_j \beta_j \]

**Pricing Risky Cashflows**

\[ P = \sum_{s} I^s X^s \]
\[ P = \frac{1}{1 + r} \sum_{s} p^s X^s \]
\[ P = \frac{1}{1 + r} \left( p^u X^u + (1 - p^u) X^d \right) \]
\[ P = \frac{1}{1 + r} E^s[X] \]
\[ p^* = I^*(1 + r) \]
\[ r_f = \frac{1}{\sum I^*} \]

Options
\[ \text{call} = \max(0, S - X) \]
\[ \text{put} = \max(0, X - S) \]
\[ P = C + \frac{X}{(1 + r)(T - t)} - S \]

Binomial Pricing
\[ \begin{align*}
p^u &= \frac{(1 + r) - d}{u - d} \\
1 - p^u &= \frac{u - (1 + r)}{u - d} \\
m &= \frac{S^d - S^u}{C^u - C^d} \\
C &= \frac{1}{1 + r} (p^u C^u + (1 - p^u) C^d) \end{align*} \]

Corporate Bonds
\[ B = \frac{1}{1 + r} D - \frac{1}{1 + r} E^* \max(D - V', 0) \]
\[ B = V - \frac{1}{1 + r} E^* \max(V' - D, 0) \]
\[ B = V - E \]

Convertible bond (special case)
\[ B = \frac{1}{1 + r} D - \frac{1}{1 + r} E^* \max(D - V', 0) + \frac{1}{1 + r} E^* \max \left( \frac{V'}{2} - D, 0 \right) \]

Corporate Finance
\[ r_E = r_V + \frac{B_L}{E_L} (r_V - r_B) \]
\[ V_L = V_U + B_L \frac{\tau_c - \tau_p}{1 - \tau_p} \]
\[ V_L = V_U + B_L \left( 1 - \frac{(1 - \tau_c)(1 - \tau_e)}{1 - \tau_p} \right) \]

\[ r_B = \frac{E_L + B_L(1 - \tau_c)}{E_L} r_{V_U} - \frac{B_L}{E_L} r_B(1 - \tau_c) \]

\[ r_{wacc} = \frac{E_L}{E_L + B_L} r_B + \frac{B_L}{E_L + B_L} r_B(1 - \tau_c) \]

Futures
\[ F_t = S_t (1 + r)^{(T-t)} \]
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Review of the First Edition

"... this is an excellent textbook in corporate finance. The book is based on a number of fundamental axioms and principles that operate as a unifying theme. A remarkable coherence is achieved between the chapters on derivative pricing and capital structure. Despite the high level of abstraction, the exposition is extremely clear, intuitive and concise. The book therefore deserves a place on the bookshelf of every finance professor and student in finance."

Journal of Finance

This course of lectures introduces students to elementary concepts of corporate finance using a more systematic approach than is generally found in other textbooks.

Axioms are first highlighted and the implications of these important concepts are studied afterwards. These implications are used to answer questions about corporate finance, including issues related to derivatives pricing, state-price probabilities, dynamic hedging, dividends, capital structure decisions, and risk and incentive management. Numerical examples are provided, and the mathematics is kept simple throughout.

In this second edition, explanations have been improved, based on the authors' experience teaching the material, especially concerning the scope of state-price probabilities in Chapter 12. There is also a new Chapter 22: Fourteen Insights.